Introduction to Measurement-Based Quantum Computation

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1 MBQC with cluster states

The discussion in section follows that of [Wei24], with the original construction from [RB01, RBB03].

1.1 Motivations

- From the perspective of implementation; good for systems with heralded noise [BR05] - go through example of probalistic entangling gates in gate model vs. cluster state model, physical systems with massive parallelism (can create large cluster states via translation-invariant interactions - e.g. neutral atoms), schemes of universal blind quantum computation [BFK10]. Also natural framework for photonic QC, where qubits can be discarded after measurement. 3D-MBQC also gives rise to a fault-tolerant way of doing QC.
- Theory perspective (my personal take); large amount of QI research is about characterizing the computational separation of classical/quantum. Non-exhaustive list of avenues of characterizing quantum-ness includes contextuality [HWVE14], wigner negativity [VFGE12], stabilizer theories [Got98], and quantum query complexity [Amb17]. MBQC gives another avenue to tackle this - single-qubit measurements + classical feedforward on initial resource state, so no entanglement is generated after initial step. Ergo - characterizing structure of resource states useful for MBQC may give insight into where the "computational power" of QC comes from. Also just being such a different computational model has inspired different ways of thinking, e.g. measurement-based simulation of gauge theories from Hiroki's talk.

1.2 Cluster states

In the circuit model, one starts with a register of n qubits in the $|0\rangle^{\otimes n}$ state, performs some arbitrary unitary via a sequence of some universal gates, and then measures at the end in the computational basis. In the measurement-based model we take a different approach. In the first step, we are either given or unitarily prepare some (entangled) universal resource state. After this, we never access unitary transformations at any point. Instead, the entire computation is carried out via single-qubit adaptive measurements on the resource state, which gets consumed as the computation proceeds.

We begin by presenting the original construction of MBQC using cluster states as the resource state. We begin by giving an operational definition of a cluster state (equivalently known as a graph state). Given a graph $G = (V, E)$ with vertices V and edges E, the cluster state $|C_G\rangle$ is defined as the state prepared by the following procedure:

1. On each vertex $v \in V$, prepare a qubit in the Pauli-X eigenstate $|+\rangle$.
2. For every pair of qubits connected by an edge $(a, b) = e \in E$, apply a controlled-Z (CZ) gate, i.e.:

$$\mathrm{CZ}_{a,b} = |0\rangle\langle 0|_a \otimes I_b + |1\rangle\langle 1|_a \otimes Z_b = \frac{I_a I_b + I_a Z_b + Z_a I_b - Z_a Z_b}{2} = I_a \otimes I_B - 2|11\rangle_{ab}\langle 11|_{ab} \tag{1.1}$$

Note that the CZ gates in the second step are symmetric and mutually commute, so they can be applied simultaneously. Physically, the second step may be accomplished via evolving the qubits under translation-invariant nearest-neighbour Ising-type interaction (independent of system size):

$$H_{\text{Ising}} = -\hbar g \sum_{(a,b)\in E} Z_a Z_b \tag{1.2}$$

This operational definition is useful for the preparation of cluster states. We now present a second, equivalent definition of the cluster state as a stabilizer state. Given a graph $G = (V, E)$, the cluster state $|C_G\rangle$ is the unique +1 eigenstate of all Pauli operators in the set:

$$\left\{ K_a = X_a \prod_{b\in \mathrm{N}(a)} Z_b | a \in V \right\} \tag{1.3}$$

where $\mathrm{N}(a) = \{b|(a,b) \in E\}$ is the neighbourhood of the vertex a. The K_a are known as the cluster state stabilizer generators.

To see that these two definitions are equivalent, we simply need to track the stabilizers of the state presented in the preparation protocol of the first definition and show that they coincide. In the first step, the stabilizer generators are simply X_a for each vertex $a \in V$. Let us see how this stabilizer evolves under a CZ gate with a neighbouring qubit:

$$X_a \mapsto (CZ_{a,b}) X_a (CZ_{a,b})^\dagger = \frac{I_a I_b + I_a Z_b + Z_a I_b - Z_a Z_b}{2} X_a \frac{I_a I_b + I_a Z_b + Z_a I_b - Z_a Z_b}{2} = X_a Z_b \tag{1.4}$$

where in the last equality we use the anticommutation of Z_a, X_a and that Paulis square to the identity. Repeating this conjugation for every edge/CZ connected to qubit a, we find that the stabilizer becomes:

$$X_a \mapsto X_a \prod_{b\in \mathrm{N}(a)} Z_b = K_a \tag{1.5}$$

the argument is analogous for any vertex $a \in V$ and hence the final cluster state has stabilizers K_a for each vertex, showing the equivalence.

This second definition is convenient for two reasons. For one, we will find for some calculations that tracking the stabilizers of the cluster state under various operations is more convenient compared to keeping track of the state ket. Second (and perhaps more importantly), the above definition allows us to view the cluster state as the unique ground state of the cluster state Hamiltonian:

$$H_{\text{cluster}} = -\sum_{a\in V} K_a. \tag{1.6}$$

This framing in terms of Hamiltonians will give rise to condensed-matter-theoretic analysis of phases of useful resource states (so called computational phases of matter), and give us a way to interpolate between a useful resource (the cluster state) and a product state. We will switch back and forth between the definitions as convenient.

1.3 Quantum wire

In the remainder of this section, we will show that adaptive measurements on cluster states provide all of the computational primitives necessary for universality. We start with the simplest primitive - the ability to move quantum information from one part of a cluster state to another, which is known as quantum wire. For this, we consider a one-dimensional cluster chain, of odd length $n+2$. We use 0-based indexing to denote the qubits. Suppose on the (leftmost) edge of the chain we have an input state $|\psi_{\text{in}}\rangle = a|0\rangle + b|1\rangle$. Then, focusing on the first two qubits, we have the state:

$$CZ_{0,1}|\psi_{\text{in}}\rangle_0 \otimes |+\rangle_1 = a|0\rangle_0|+\rangle_1 + b|1\rangle_0|-\rangle_1 = |+\rangle_0 \left(a|+\rangle_1 + b|-\rangle_1\right) + |-\rangle_0 \left(a|+\rangle_1 - b|-\rangle_1\right) \tag{1.7}$$

If we measure the first qubit in the X-basis and obtain outcome s_0 (taking $s_0 = 0$ if we measure the $+1$ eigenvalue and $s_0 = 1$ if we measure the -1 eigenvalue), the resulting state on the second qubit is:

$$a|+\rangle_1 + bX^{s_0}|-\rangle_1 = HZ^{s_0}(a|0\rangle_1 + b|1\rangle_1) = HZ^{s_0}|\psi_{\text{in}}\rangle_1 \tag{1.8}$$

In other words, up to the operator HZ^{s_0} the X-measurement on the first qubit has teleported the input state $|\psi_{\text{in}}\rangle$ to the second qubit. The HZ^{s_0} is known as a byproduct operator and can be kept track of classically to correct the output state at the end. This modification on the quantum teleportation protocol [BBC+93], called "half-teleportation" allows us to wire quantum information from one part of the cluster state to another[1]. We can repeat this process along the $n+2$-qubit chain, measuring all qubits in the X basis save for the rightmost qubit; the output state on this last qubit is simply the output of $n+1$ half-teleportations:

$$\begin{aligned} HZ^{s_n}HZ^{s_{n-1}}\ldots HZ^{s_2}HZ^{s_1}HZ^{s_0}|\psi_{\text{in}}\rangle_{n-1} &= X^{s_n}Z^{s_{n-1}}\ldots Z^{s_2}X^{s_1}Z^{s_0}|\psi_{\text{in}}\rangle_{n-1} \\ &\cong X^{\sum_{i,\text{odd}} s_i}Z^{\sum_{i,\text{even}} s_i}|\psi_{\text{in}}\rangle_{n-1} \end{aligned} \tag{1.9}$$

where we note we have used the irrelevancy of global phase to combine the X and Z operators. Up to the total byproduct operator $X^{\sum_{i,\text{odd}} s_i}Z^{\sum_{i,\text{even}} s_i}$ which can be classically tracked and applied, we have successfully teleported the initial quantum state from one end of the chain (qubit 0) to the other (qubit $n+1$).

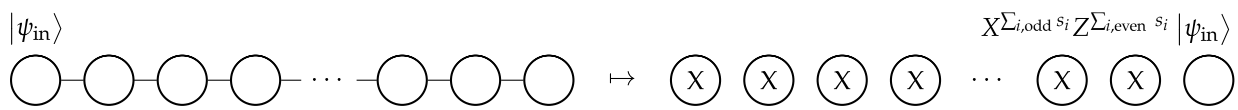


Figure 1.1: Pictorial depiction of quantum teleportation on a cluster chain. A linear cluster state is constructed with $|\psi_{\text{in}}\rangle$ as a logical input qubit state on the first qubit. By measuring all the qubits (save the last) in the X-basis, up to a measurement outcome-dependent byproduct operator, $|\psi_{\text{in}}\rangle$ is teleported to the final qubit.

1.4 Single-qubit operations

We now have the ability to teleport our quantum state - the MBQC equivalent of the identity operation. We now expand our toolkit to logical evolutions of our encoded single-qubit state. To this end, suppose we wanted to perform the Z-rotation $RZ(\beta)$. One way would be to input $RZ(\beta)|\psi_{\text{in}}\rangle$ instead of $|\psi_{\text{in}}\rangle$, and then the output from the previous part would be $HZ^{s_0}RZ(\beta)|\psi_{\text{in}}\rangle$. This however would contradict the central idea of MBQC, that we don't have access to unitary operations beyond the initial construction of the resource state. But, let us see if we can accomplish something equivalent with the single-qubit measurements that we do have. First, note that we can commute the z-rotation across the CZ gate:

$$\Pi^0_{\pm}CZ_{0,1}RZ^0(\beta)|\psi_{\text{in}}\rangle_0|+\rangle_1 = \Pi^0_{\pm}RZ^0(\beta)CZ_{0,1}|\psi_{\text{in}}\rangle_0|+\rangle_1 \tag{1.10}$$

Now, let us see how the z-rotation modifies the X-basis measurement $\Pi_{\pm}$:

$$\Pi_{\pm}RZ(\beta) = \frac{I \pm X}{2}RZ(\beta) = RZ(\beta)RZ(-\beta)\frac{I \pm X}{2}RZ(\beta) \cong \frac{I \pm RZ(-\beta)XRZ(\beta)}{2} \tag{1.11}$$

where we use that $RZ(\beta)RZ(-\beta) = I$ and discard the final rotation in the last step by noting that unitary operations performed after the measurement are irrelevant. We see from the above that by measuring not in the X-basis but rather in the rotated basis of the observable:

$$O(\beta) = RZ(-\beta)XRZ(\beta) = \cos(\beta)X - \sin(\beta)Y \tag{1.12}$$

[1]Those with a background in quantum mechanics are likely unsurprised by the ability for cluster states to carry out quantum wire, as the 2-qubit cluster state is nothing more than a 2-qubit Bell pair (up to a local Hadamard) which is the standard resource in the quantum teleportation protocol.

the logical state is not only teleported, but is rotated by $RZ(\beta)$, giving us the output of $HZ^{s_0}RZ(\beta)|\psi_{\text{in}}\rangle$ as desired.

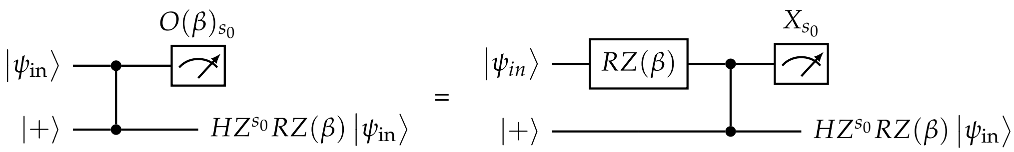


Figure 1.2: Z-rotation circuit identity. Measuring in the rotated basis of $O(\beta) = RZ(-\beta)XRZ(\beta)$ is logically identical to applying a Z-rotation on the input followed by a half-teleportation of the state. Therefore, we can implement a logical Z-rotation via choice of the measurement basis. Figure adapted from [Wei22].

To go from just Z-rotations to arbitrary rotations, we require the ability to perform a general rotation $R(\beta_0, \beta_1, \beta_2) = RZ(\beta_2)RX(\beta_1)RZ(\beta_0)$, invoking the Euler decomposition of rotations. To this end, we consider a 4-qubit cluster chain where we measure the first qubit in $O(\beta_0)$, the second qubit in $O(\beta_1)$, and the third qubit in $O(\beta_2)$. This amounts to three instances of the above procedure, so the quantum state of the fourth/output qubit will be:

$$HZ^{s_2}RZ(\beta_2)HZ^{s_1}RZ(\beta_1)HZ^{s_0}RZ(\beta_0)|\psi_{\text{in}}\rangle_3 = HZ^{s_2}RZ(\beta_2)X^{s_1}RX(\beta_1)Z^{s_0}RZ(\beta_0)|\psi_{\text{in}}\rangle_3 \tag{1.13}$$

Moving the byproduct operators to the left, we obtain:

$$HZ^{s_2+s_0}X^{s_1}RZ((-1)^{s_1}\beta_2)RX((-1)^{s_0}\beta_1)RZ(\beta_0)|\psi_{\text{in}}\rangle_3 \tag{1.14}$$

where we see that the byproduct operators do not just induce an irrelevant global phase but also flip the rotation angles. This at first seems problematic - the measurement outcomes are random and this poses obstacle to the deterministic implementation of $RZ(\beta_2)RX(\beta_1)RZ(\beta_0)$. However, this is not the case. We can adaptively choose future measurement bases conditioned on the past measurement outcomes, namely choosing the measurement basis of $O((-1)^{s_0}\beta_1)$ conditioned on the outcome of the first qubit and the basis of $O((-1)^{s_1}\beta_2)$ conditioned on the outcome of the second qubit. This deterministically yields the output:

$$HZ^{s_2+s_0}X^{s_1}RZ(\beta_2)RX(\beta_1)RZ(\beta_0)|\psi_{\text{in}}\rangle_3 \tag{1.15}$$

as desired. We thus are able to perform arbitrary single-qubit unitaries via adaptive single-qubit measurements on the (linear) cluster state, and thus the cluster chain is universal for a single-qubit quantum computation. This adaptivity is the key ingredient that makes MBQC deterministic despite the probabilistic nature of quantum measurement, and is what gives rise to temporal order within MBQC.

To spell out how we got to the output state in Eq. (1.15) in more detail, starting from a 4-qubit cluster chain with $|\psi_{\text{in}}\rangle$ on the input qubit of 0; measuring qubit 0 in $O(\beta_0)$ rotates and teleports the state, yielding (on qubit 1)

$$HZ^{s_0}R_z(\beta_0)|\psi_{\text{in}}\rangle_1 \tag{1.16}$$

Then measuring qubit 1 in $O((-1)^{s_0}\beta_1)$, we get on qubit 2:

$$HZ^{s_1}R_z((-1)^{s_0}\beta_1)HZ^{s_0}R_z(\beta_0)|\psi_{\text{in}}\rangle_2 \tag{1.17}$$

finally measuring qubit 2 in $O((-1)^{s_1}\beta_2)$, we get on qubit 3:

$$HZ^{s_2}R_z((-1)^{s_1}\beta_2)HZ^{s_1}R_z((-1)^{s_0}\beta_1)HZ^{s_0}R_z(\beta_0)|\psi_{\text{in}}\rangle_3 \tag{1.18}$$

Now the conjugation with the Hadamards means this is equivalent to:

$$HZ^{s_2}R_z((-1)^{s_1}\beta_2)X^{s_1}R_x((-1)^{s_0}\beta_1)Z^{s_0}R_z(\beta_0)|\psi_{\text{in}}\rangle_3 \tag{1.19}$$

Now moving the byproduct operators to the left, the Z^{s_0} flips the β_1 x-rotation sign and the X^{s_1} flips the β_2 z-rotation sign:

$$HZ^{s_2}X^{s_1}Z^{s_0}R_z((-1)^{s_1}(-1)^{s_1}\beta_2)X^{s_1}R_x((-1)^{s_0}(-1)^{s_0}\beta_1)Z^{s_0}R_z(\beta_0)|\psi_{\text{in}}\rangle_3 \tag{1.20}$$

$(-1)^2 = 1$ and we can move past Z^{s_0} modulo a global phase, so we get (as claimed):

$$HZ^{s_2+s_0}X^{s_1}RZ(\beta_2)RX(\beta_1)RZ(\beta_0)|\psi_{\text{in}}\rangle_3 \tag{1.21}$$

1.5 Two-qubit operations and universality

We now upgrade from a cluster chain to a two-dimensonal cluster state defined on a square grid. We saw how the action of X-measurements on a cluster state to teleport quantum information. Z measurements play a complementary role in MBQC - namely, they slice out qubits in the cluster. Operationally, this allows us to separate out a two-dimensional cluster into chains corresponding to different logical qubits, as demonstrated by Fig. 1.3. To each of these logical qubits, we may use the constructions of the previous section to perform arbitrary single-qubit logical operations.

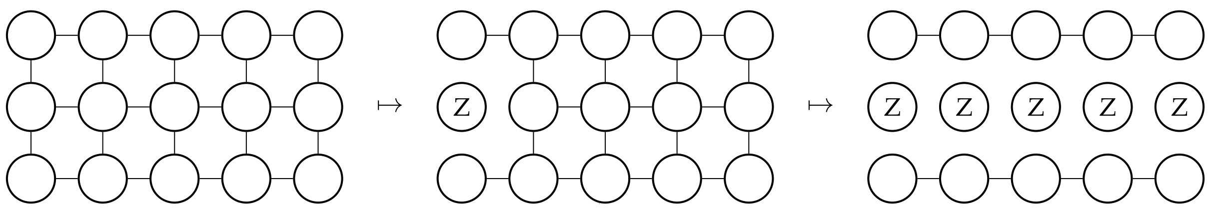


Figure 1.3: Z-measurements on a cluster state. In the central panel, we measure the leftmost center qubit in the Z basis, disconnecting it from the rest of the cluster. In the rightmost panel, we measure all central qubits in the Z basis, separating the cluster state into two cluster chains, each which correspond to a logical qubit. This procedure can be generalized to separate out a height $2k-1$ rectangular cluster state into k logical qubits.

The slicing property of Z-measurements is most easily seen by looking at the stabilizers of the cluster state. Measuring in the Z-basis on qubit a enforces a new stabilizer $\pm Z_a$. This commutes with all cluster state stabilizers save for the one centered on a, i.e. $K_a = X_a \prod_{b\in N(a)} Z_b$. This one stabilizer is removed/replaced by Z_a, and thus this qubit is removed/disentangled from the overall cluster state (the rest of which is left intact). Hence, Z-measurements allow us to separate regions of a given cluster state. By extending the height of the cluster state and separating chains, we gain the ability to simulate an arbitrary number of logical qubits.

Given a sufficiently large graph state, we now possess the ability to simulate an arbitrary number of qubits and single-qubit operations on such qubits. To promote our scheme to universality, we require one more operation, namely the ability to carry out two-qubit entangling gates. Here we present the simplest such construction, namely a controlled-NOT (CNOT) gate between two logical qubits using 4 physical qubits of the cluster. This is graphically depicted in Fig. 1.4.

We start with input logical qubit states $|\psi^c\rangle_1 = a|0\rangle_1 + b|1\rangle_1$ and $|\psi^t\rangle_2 = c|0\rangle_2 + d|1\rangle_2$ that we wish to apply a logical CNOT between. Physical qubits 3/4 in the above protocol are intialized to the $|+\rangle$ state. Applying CZ gates between $1/3, 2/3, 3/4$ we obtain the state:

$$\begin{aligned}&CZ_{13}CZ_{23}CZ_{34}|\psi^c\rangle_1|\psi^t\rangle_2|+\rangle_3|+\rangle_4\\ =&(a|0\rangle_1 + b|1\rangle)(c|0\rangle_2 + d|1\rangle_2)|0\rangle_3|+\rangle_4 + (a|0\rangle_1 - b|1\rangle)(c|0\rangle_2 - d|1\rangle_2)|1\rangle_3|-\rangle_4\end{aligned} \tag{1.22}$$

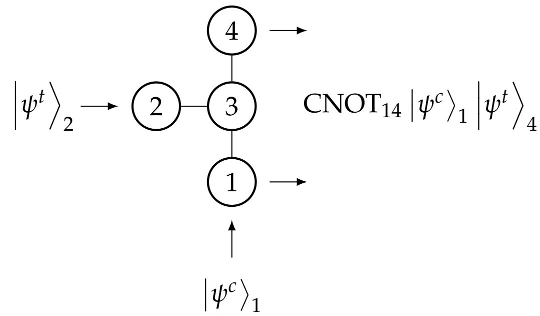


Figure 1.4: Realization of logical CNOT gate in MBQC. We consider the following subsection of the cluster state composed of 4 physical qubits. On qubits 1/2 we have the input states $|\psi^c\rangle_1$ and $|\psi^t\rangle_2$. Qubits 3/4 start in the $|+\rangle$ state. A CZ gate is applied between qubit 3 and 1/2/4. Then by measuring qubits 2/3 in the X-basis, the resulting output state on qubits 1/4 (up to byproduct operator $Z_1^{s_2}X_4^{s_3}Z_4^{s_2}$ depending on the measurement outcomes) is $\text{CNOT}_{14}|\psi^c\rangle_1|\psi^t\rangle_4$.

where we treat qubit 3 as the control qubit. Rewriting the above, we find:

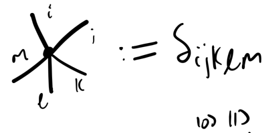
$$\begin{aligned}
&\frac{1}{2}|{+}{+}\rangle_{23}\left(ac|00\rangle_{14}+ad|01\rangle_{14}+bd|10\rangle_{14}+bc|11\rangle_{14}\right)\\
+&\frac{1}{2}|{+}{-}\rangle_{23}\left(ad|00\rangle_{14}+ac|01\rangle_{14}+bc|10\rangle_{14}+bd|11\rangle_{14}\right)\\
+&\frac{1}{2}|{-}{+}\rangle_{23}\left(ac|00\rangle_{14}-ad|01\rangle_{14}-bd|10\rangle_{14}+bc|11\rangle_{14}\right)\\
+&\frac{1}{2}|{-}{-}\rangle_{23}\left(-ad|00\rangle_{14}+ac|01\rangle_{14}+bc|10\rangle_{14}-bd|11\rangle_{14}\right)\\
=&\frac{1}{2}|{+}{+}\rangle_{23}\text{CNOT}_{14}|\psi^c\rangle_1|\psi^t\rangle_4+\frac{1}{2}|{+}{-}\rangle_{23}X_4\text{CNOT}_{14}|\psi^c\rangle_1|\psi^t\rangle_4\\
+&\frac{1}{2}|{-}{+}\rangle_{23}Z_1Z_4\text{CNOT}_{14}|\psi^c\rangle_1|\psi^t\rangle_4+\frac{1}{2}|{-}{-}\rangle_{23}Z_1X_4Z_4\text{CNOT}_{14}|\psi^c\rangle_1|\psi^t\rangle_4
\end{aligned} \tag{1.23}$$

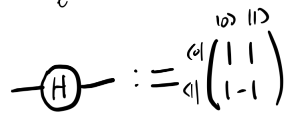
from which we conclude that the output on qubits 1/4 after the measurements is $Z_1^{s_2}X_4^{s_3}Z_4^{s_2}\text{CNOT}_{14}|\psi^c\rangle_1|\psi^t\rangle_4$. Up to measurement-outcome dependent byproduct operators, we obtain the claimed logical CNOT between qubits.

Thus, with arbitrary single unitaries and the CNOT, we have an universal gate set. Given a sufficiently tall (to give us a sufficient number of logical qubits) and long (to give us sufficient circuit depth) cluster state we are therefore able to perform an arbitrary quantum computation. Thus single-qubit measurements on a two-dimensional cluster state is a universal model of quantum computation[2]. We also note that going to one higher dimension provides us with further phenomenology, with the jump to three-dimensional cluster states providing a method to perform MBQC fault-tolerantly [RHG06, RHG07].

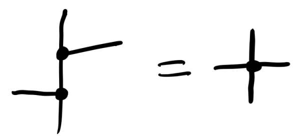
Before moving onto SPT-MBQC, we note that the argument for the *CNOT* gate we did above can be simplified by thinking about cluster states in a graphical way using tensors (this picture is also going to be very natural in the next section, when we formally introduce the notion of matrix product states). Let us walk through the argument. We define the delta and Hadamard tensors (note: throughout, we neglect the normalization for simplicity):

[2] We note also of the existence of a construction [SHW+22] that uses the machinery of dual-unitary circuits to accomplish universal MBQC in one spatial dimension.

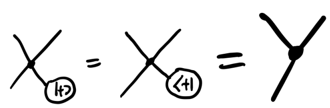




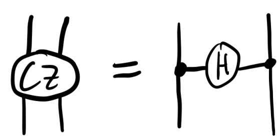
In preparation for arguing the CNOT gate with this graphical calculus, we first introduce some simple identities. First, note that delta tensors compose (as the product of two deltas is another delta):



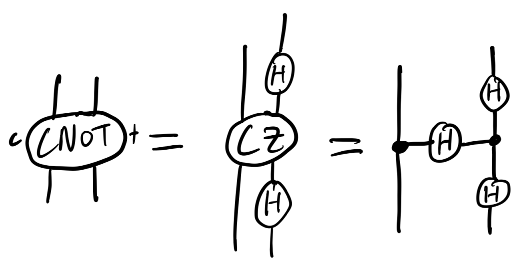
Since $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, when it (or $\langle+|$) hits a delta tensor it can simply be removed:



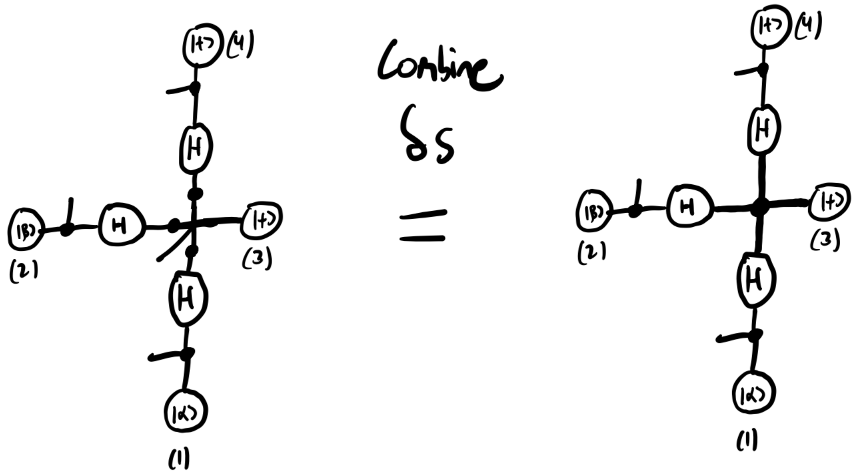
A CZ gate can graphically be written as the below, noting that the Hadamard only fires to give the minus sign when both legs have 1:

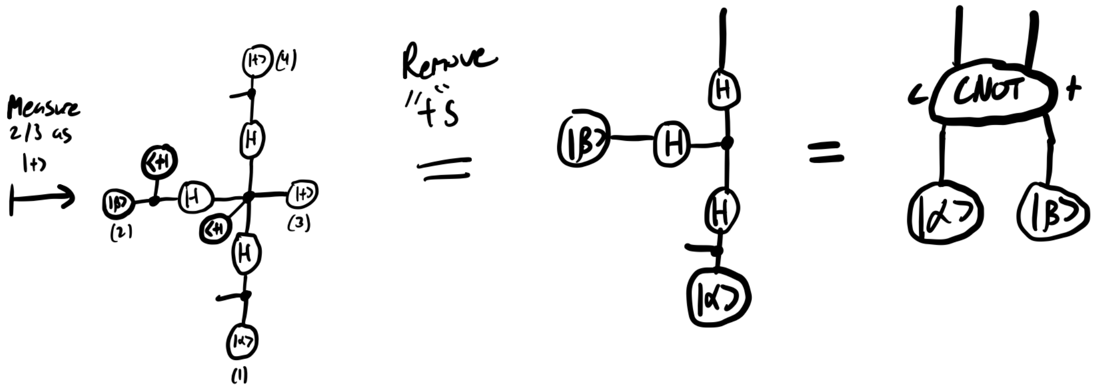


The $CNOT$ gate is then implemented by conjugating one of the two qubits using Hadamards:



Combining these identities, we have the following argument for the logical CNOT gate, in graphical form:





Though I'm a little uncertain on why some of the delta tensors seem to be dropped here.

2 SPT (Symmetry-Protected Topological) MBQC

Natural question; are cluster states the only useful states as MBQC resources? Resource states are actually exponentially rare in Hilbert space [GFE09] - perhaps counterintuitively, too entangled to be useful in such a scheme (note that cluster state is SRE - only takes $O(1)$ bounded-depth gates to prepare). This is elucidated in the presence of symmetry - computational power is uniform within SPTs.

2.1 Lightning Intro to MPS

MPS; consider N-qudit state:

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_N}^{d} c_{i_1, \ldots, i_N} |i_1, \ldots, i_N\rangle \tag{2.1}$$

Store amplitudes in a $d_1 \times d_2 \times \ldots \times d_N$ tensor (matrix). No gain yet - but, we can consider the ansatz of a tensor network - MPS in 1D (for simplicity):

$$c_{i_1, i_2, \ldots, i_N} = \sum_{\alpha_1, \ldots, \alpha_N = 1}^{D} A^{i_1}_{\alpha_1, \alpha_2} A^{i_2}_{\alpha_2, \alpha_3} \ldots A^{i_N}_{\alpha_N, \alpha_1} \tag{2.2}$$

With:

$$A^i = \sum_{\alpha,\beta=1}^{D} A^i_{\alpha,\beta} |\alpha\rangle\langle\beta| \tag{2.3}$$

so:

$$c_{i_1,\ldots,i_N} = \mathrm{Tr}[A^{i_1} A^{i_2} \ldots A^{i_N}] \tag{2.4}$$

Generically, $D \approx d^{N/2}$ but for gapped MPS following area law (remember - SPT states are SRE, so in some sense we expect that they will be able to be efficiently represented) states, $D \sim \mathrm{poly}(N)$. Obtained via Schmidt decomposition across sites ($|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$), Schmidt rank is the bond dimension, or measure of entanglement. Allows us to simplify to dD^2N instead of exponentially many d^N coefficients, as the Schmidt coefficeints decay quickly to allow us to discard many of them. This way of thinking is useful not just conceptually but also for simulation, e.g. DMRG algorithm.

Note for a system with boundaries (as will soon become relevant for MBQC on a chain...)

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N}^{d} \langle L| A^{i_1} A^{i_2} \ldots A^{i_N} |R\rangle |i_1,\ldots,i_N\rangle \tag{2.5}$$

with BC specified by the virtual $|L\rangle, |R\rangle$.

Example: GHZ:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}) = \tag{2.6}$$

Has:

$$A^0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A^1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

for each site $1, \ldots N$.

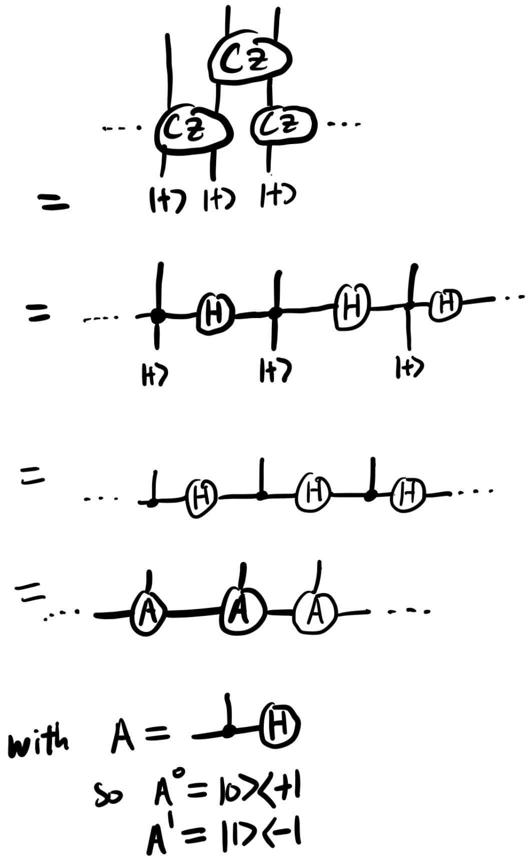
Example; cluster state. Using definition of CZs applied to all neighbouring $|+\rangle$ states on chain, can write as:

$$|C\rangle_N = \frac{1}{2^{N/2}} \sum_{i_1,\ldots,i_N} (-1)^{\text{\# neighbouring 1s}} |i_1,\ldots,i_N\rangle \tag{2.8}$$

Which then invites:

$$A^0 = |0\rangle\langle +|, \quad A^1 = |1\rangle\langle -| \tag{2.9}$$

The -1 only fires when two A^1s are next to each other! Can also derive this graphically:



Wire/Symmetry-respecting/X-basis is convenient to think about. Consider representation of the above in this basis. To this end, we consider:

$$A[b] := \sum_i \langle b|i\rangle A^i \tag{2.10}$$

I.e. the contraction the physical leg/index with the basis state $|b\rangle$. Where then the tensors can be representated as:

$$A[+] = \langle +|0\rangle A^0 + \langle +|1\rangle A^1 = \frac{1}{\sqrt{2}}(A^0 + A^1) = H \tag{2.11}$$

$$A[-] = \langle -|0\rangle A^0 + \langle -|1\rangle A^1 = \frac{1}{\sqrt{2}}(A^0 - A^1) = HZ \tag{2.12}$$

If we instead look at the rotated basis $\big\{|+\beta\rangle = R_z(\beta)|+\rangle, |-\beta\rangle = R_z(\beta)|-\rangle\big\}$

$$A[+\beta] = \langle +\beta|0\rangle A^0 + \langle +\beta|1\rangle A^1 = HR_z(\beta) \tag{2.13}$$

$$A[-\beta] = \langle -\beta|0\rangle A^0 + \langle -\beta|1\rangle A^1 = HZR_z(\beta) \tag{2.14}$$

2.2 MBQC in correlation space

MPS gives us another way of thinking about MBQC, will be useful for showing why SPTs are computational phases. Evolution of logical qubit in MBQC corresponds to evolution of virtual/correlation space left edge qubit:

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N}^{d} \langle L|A^{i_1}A^{i_2}\ldots A^{i_N}|R\rangle|i_1,\ldots,i_N\rangle \mapsto \sum_{i_2,\ldots,i_N}^{d} \langle L'|A^{i_2}\ldots A^{i_N}|R\rangle|i_1,\ldots,i_N\rangle \tag{2.15}$$

where:

$$|L'\rangle = A^\dagger[k]|L\rangle \tag{2.16}$$

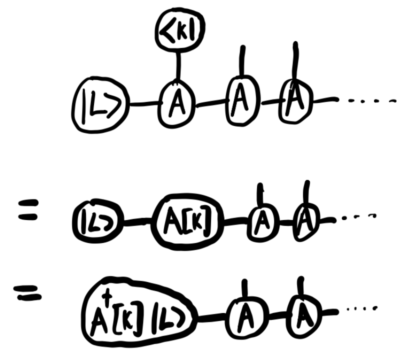
where k is the measurement outcome corresponding to the measured observable. So; consider a wire/X basis measurement, with outcome k. Then:

$$|L'\rangle = HZ^k|L\rangle \tag{2.17}$$

with the state $|L\rangle$ teleported site to the right. This is exactly quantum wire (up to byproducts). Same can be done with rotated basis measurement:

$$|L'\rangle = A^\dagger[k]|L\rangle = HZ^k R_z(\beta)|L\rangle \tag{2.18}$$

Graphically, the correlation space evolution of the qubit looks like:



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