The Physics of Snowmageddon

A back-of-the-envelope calculation for a very chilly Earth

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UBC Physics Circle

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Vancouver Snowmageddon 2021

Image credit: Shutterstock

A snow-covered world

Image credit: Snowpiercer film, Simon De Salvatore

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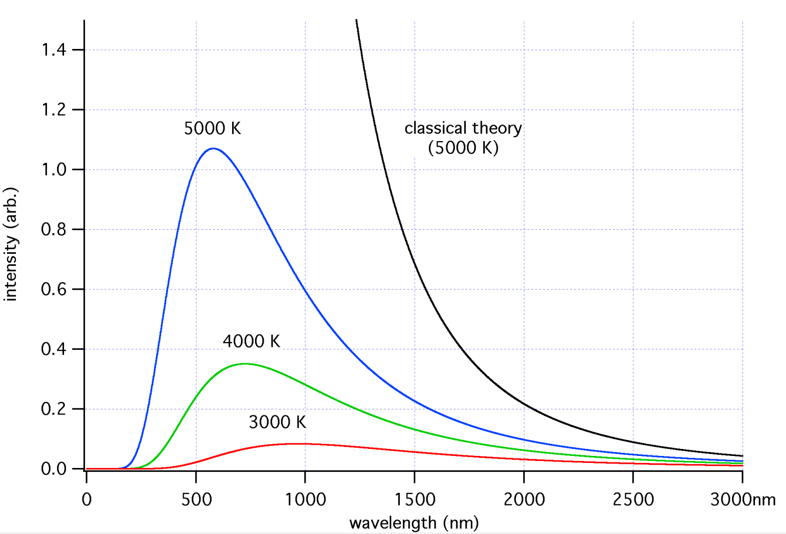
$$P_{in} = P_{out} \tag{1}$$

- P is power, and represents energy per unit time.

Image credit: Snowpiercer film, Simon De Salvatore

The Blackbody Spectrum



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Radiation from the Sun

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$$P = \varepsilon \sigma A T^4. \tag{2}$$

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- This is known known as the *Stefan–Boltzmann law*. Since the sun emits radiation like a perfect blackbody, $\varepsilon = 1$ (and we can assume this for Earth as well) and so:

$$P = \sigma A T^4 \tag{3}$$

Estimating σ - What constants?

- Dimensional analysis!

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- First, what universal constants would naturally go into σ?

Defining constant	Symbol	Units	"Purpose"
Hyperfine transition freq. of Cs	$\Delta\nu_{\mathrm{Cs}}$	s^{-1}	Fundamental time unit
Speed of light	c	$\mathrm{m\,s}^{-1}$	Universal speed limit/speed of photons
(Reduced) Planck's constant	$\hbar$	$\mathrm{J\,s}$	Defines small-scale energy
Elementary charge	e	C	Electron/proton charge
Boltzmann constant	k_B	$\mathrm{J\,K}^{-1}$	Energy-temperature conversion factor
Avogadro constant	N_A	mol^{-1}	Molecules per mole

Estimating σ - What powers?

- So, we've identified the relevant fundamental constants to be c, $\hbar$, and k_B. Can we combine them to figure out what σ is given what units/dimensions we want?

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$$P = \sigma A T^4$$

Symbol	Units	Dimensions
c	$\mathrm{m\,s^{-1}}$	$[L][T]^{-1}$
$\hbar$	$\mathrm{J\,s}$	$[E][T]$
k_B	$\mathrm{J\,K^{-1}}$	$[E][\Theta]^{-1}$

- $[L]$ - length, $[T]$ - time, $[\Theta]$ - temperature, $[E]$ - Energy.

Estimating σ - Solution

- P has units $\mathrm{W} = \mathrm{J\,s^{-1}}$, T^4 has units K^4, and A has units m^2, so for the dimensions to be consistent, we would have:

$$\sigma \sim \frac{k_B^4}{c^2 \hbar^3} \tag{4}$$

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- The actual value turns out to be not so far off! Only off by a factor of 1.4 or so.

$$\sigma = \frac{2\pi^2}{15} \frac{k_B^4}{c^2 \hbar^3} \approx 5.67 \times 10^{-8} \mathrm{W\,m^{-2}\,K^{-4}} \tag{5}$$

- Let T_S be the temperature of the sun's surface, R_S be the radius of the sun, R_E be the radius of the Earth, and D be the distance between the sun and Earth.
- The power of solar radiation at the sun's surface is...
    1. $P_S = \sigma(4\pi R_S^2)T_S^4$
    2. $P_S = \sigma(4\pi R_E^2)T_S^4$
    3. $P_S = \sigma(4\pi D^2)T_S^4$
    4. $P_S = \sigma(\pi R_S^2)T_S^4$
    5. $P_S = \sigma(\pi R_E^2)T_S^4$
    6. $P_S = \sigma(\pi D^2)T_S^4$

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How much energy from the sun? - Solar Power (solution)

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How much energy from the sun? - Energy shell

- Let T_S be the temperature of the sun's surface, R_S be the radius of the sun, R_E be the radius of the Earth, and D be the distance between the sun and Earth.
- By the time this energy reaches the Earth, it is spread out over an area...
  1. $4\pi R_S^2$
  2. $4\pi R_E^2$
  3. $4\pi D^2$
  4. πR_S^2
  5. πR_E^2
  6. πD^2

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Image credit: Shutterstock

- Let T_S be the temperature of the sun's surface, R_S be the radius of the sun, R_E be the radius of the Earth, and D be the distance between the sun and Earth.
- How much area of the solar radiation energy shell (from the previous question) does Earth take up? (Hint: What is the area of a shadow if I shine a light straight above a sphere?)
    1. $4\pi R_S^2$
    2. $4\pi R_E^2$
    3. $4\pi D^2$
    4. πR_S^2
    5. πR_E^2
    6. πD^2

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How much energy from the sun? - Effective Earth Area (solution)

- Let T_S be the temperature of the sun's surface, R_S be the radius of the sun, R_E be the radius of the Earth, and D be the distance between the sun and Earth.
- How much area of this solar radiation energy shell does Earth take up? (Hint: What is the area of a shadow if I shine a light straight above a sphere?)
  1. $4\pi R_S^2$
  2. $4\pi R_E^2$
  3. $4\pi D^2$
  4. πR_S^2
  5. $\boxed{\pi R_E^2}$
  6. πD^2

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How much power do we get from the sun? - Conclusion

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- This energy is spread out over a shell of area $4\pi D^2$.
- Earth takes up πR_E^2 of this shell.
- Combining this, we get:

$$P_{in} = P_S \frac{\pi R_E^2}{4\pi D^2} = \frac{\sigma \pi R_S^2 T_S^4 R_E^2}{D^2} \tag{6}$$

Albedo - What is it?

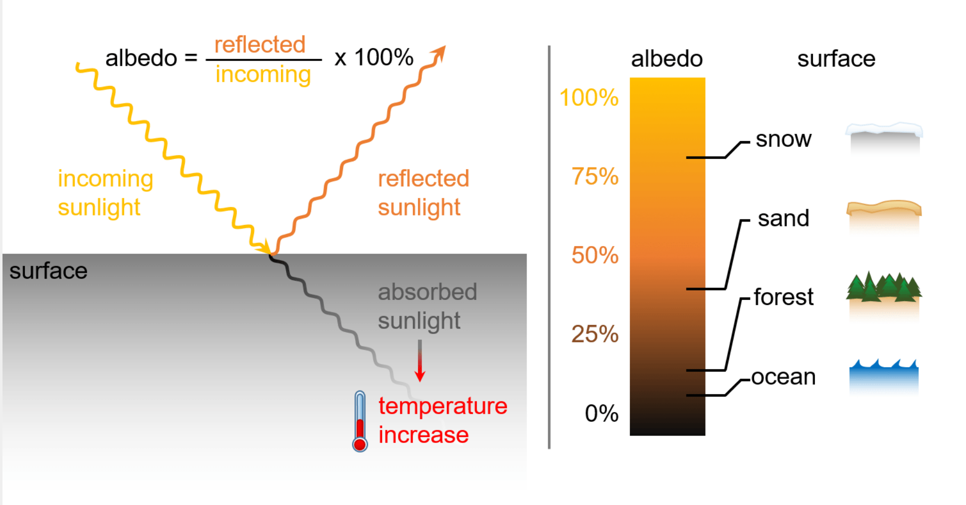


Image credit: Harvard University

Albedo - Why is α for snow so high?

- Why does so much light reflect off of snow?

Image credit: Colorado State University

Albedo - Why is α for snow so high?

- Why does so much light reflect off of snow?
- Collection of lots of small crystals $\Longrightarrow$ lots of oppurtunities for reflection!

Image credit: Colorado State University

So, we have...

- $\alpha \approx 0.8$ for snow.
- $\alpha \approx 0.3$ for the current Earth.
- Refining our earlier equation to account for albedo, we have:

$$P_{in} = (1 - \alpha)\frac{\sigma\pi R_S^2 T_S^4 R_E^2}{D^2} \tag{7}$$

Putting it together

1. Estimate the steady state temperature T_E of the snow-covered Earth ($\alpha = 0.8$) with our model. Is your result reasonable?
2. Estimate the steady state temperature of the current (non-snow covered) Earth ($\alpha = 0.3$) with our model. How does compare to the actual value of 288K? How could we improve our model?

$$P_{in} = P_{out}$$

$$P_{in} = (1-\alpha)\frac{\sigma\pi R_S^2 T_S^4 R_E^2}{D^2}$$

$$P_{out} = A_E \sigma T_E^4$$

$$D = 1.47 \times 10^{11}\text{m}$$

$$R_E = 6.37 \times 10^{6}\text{m}$$

$$R_S = 6.96 \times 10^{8}\text{m}$$

$$T_S = 5800\text{K}$$

$$\sigma = 5.67 \times 10^{-8}\,\text{W}\,\text{m}^{-2}\,\text{K}^{-4}$$

Combining the first three equations, we get:

$$A_E \sigma T_E^4 = (1-\alpha)\frac{\sigma \pi R_S^2 T_S^4 R_E^2}{D^2} \tag{8}$$

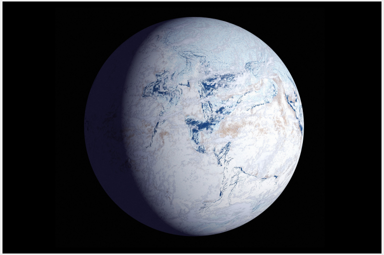
The earth has surface area $A_E = 4\pi R_E^2$, so:

$$4\pi R_E^2 \sigma T_E^4 = (1-\alpha)\frac{\sigma \pi R_S^2 T_S^4 R_E^2}{D^2} \tag{9}$$

Solving for T_E and cancelling out like terms, we get:

$$T_E = \sqrt[4]{\frac{(1-\alpha) R_S^2 T_S^4}{4D^2}} \tag{10}$$

Plugging in numbers for R_s, T_S, and D, we get $T_E \sim 188\mathrm{K}$ for the snowball Earth and $T_E \sim 258\mathrm{K}$ for the current Earth.



	Predicted Temp.	Actual Temp.
Snowball Earth	188K/-85C°	223K/ -50C°
Current Earth	258K/-15C°	288K/15C°

Comparing the predicted versus the actual values, we observe an underestimate. One reason is we didn't take into account the atmosphere and how it acts to retain heat. Another possible consideration for our model is that ice has lower albedo than snow, and it might be unrealistic for the total ice-age Earth to be covered completely in snow (some/much of it may be glacial ice).

Image credit: Science Source