Physics Circle

Symmetry in Electrostatics

January 28, 2021

The Physicist's Toolbox (incomplete)

- Dimensional Analysis (Last year)
- Fermi Approximations (Last year)
- Simulations/Predicting the future (Last session!)
- Scaling Analysis (Maybe later?)
- **Symmetry Arguments** (Today!)

Why Symmetry?

- Applicable in nearly every field of physics
- Gives rise to fundamental physical laws & structure
- Example: Noether's Theorem
- Can make solving problems a lot easier! $\rightarrow$ Today, we will exploit symmetries to solve hard electrostatics problems!

Background - Electric Field

The Coulomb force

Experimentally, the force from a charge q_1 onto a charge q_2 is given by:

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathcal{R}^2}\hat{\mathcal{R}} \tag{1}$$

where r is the distance between the charges, and $\hat{\mathcal{R}}$ is a vector that points from one charge to the other.

Electric Field

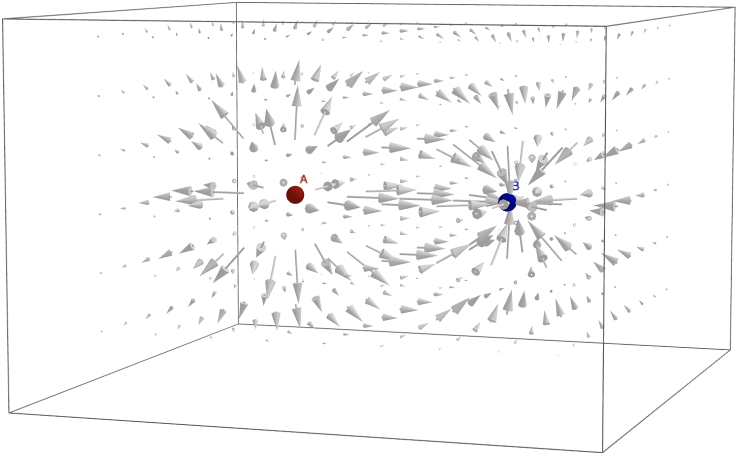
A point charge q_1 generates an electric field:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{r} \tag{2}$$

And the force that a charge q_2 feels in an electric field is:

$$\vec{F} = q_2\vec{E} \tag{3}$$

Background - Electric Field



You can play around with the simulation here: https://www.geogebra.org/m/qgQF6NtC. Notice how the electric field points from positive to negative.

Background - Superposition

Superposition of Electric Fields

If we have electric fields $\overrightarrow{E}_1, \overrightarrow{E}_2, \cdots, \overrightarrow{E}_n$, the total electric field is given by:

$$\overrightarrow{E}_{tot} = \overrightarrow{E}_1 + \overrightarrow{E}_2 + \cdots + \overrightarrow{E}_n \tag{4}$$

Background - Superposition

Superposition of Electric Fields

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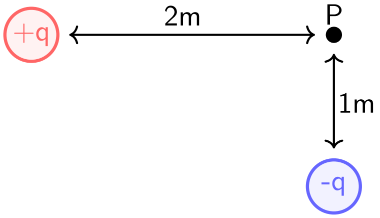
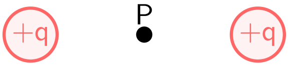
$$\overrightarrow{E}_{tot} = \overrightarrow{E}_1 + \overrightarrow{E}_2 + \cdots + \overrightarrow{E}_n \quad (4)$$

Example: The electric field at point P can be calculated by the sum of the two contributions:



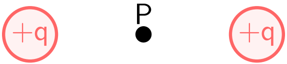
1a. Charged Shapes



Q: Suppose I place positive charges $+q$ at two ends of a line. What is the electric field at the center (point P)? Equivalently, If I were to place a charge at point P, would it feel a net electric force?

1. It's nonzero
2. It's zero

1a. Charged Shapes



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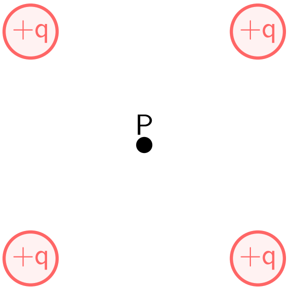
1. It's nonzero
2. It's zero

A: The electric field is zero at the center as the contributions from the two charges cancel by symmetry.

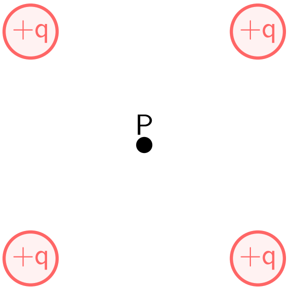
1b. Charged shapes



Q: Suppose I place positive charges $+q$ at the four corners of a square. What is the electric field at the center (point P)?

a. It's nonzero
b. It's zero

1b. Charged shapes



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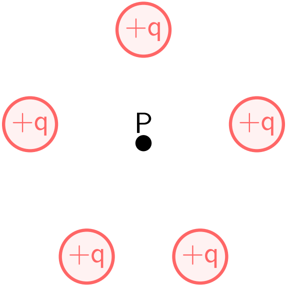
(a) It's nonzero

(b) It's zero

A: The electric field is zero by symmetry; the electric field from each charge is perfectly cancelled out from the charge opposite from it.

1c. Charged shapes



Q: Now I suppose I place positive charges Q at each of the vertices of a **pentagon**. What is the electric field at the center?

(a) It's nonzero

(b) It's zero

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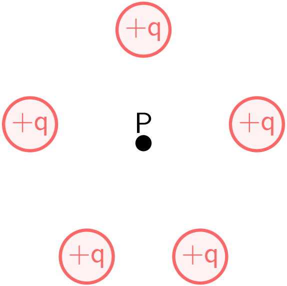

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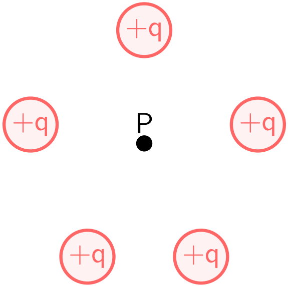
A: The electric field is again zero by symmetry; in fact this result holds for charges distributed on the corners of any n-gon!

1c. Charged shapes



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(a) It's nonzero

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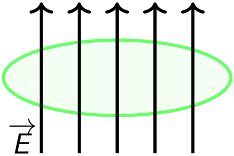
A: The electric field is again zero by symmetry; in fact this result holds for charges distributed on the corners of any n-gon!
(Bonus Problem: What would be the electric field at the center if we were to remove the charge at the top vertex?)

Background - Electric Flux

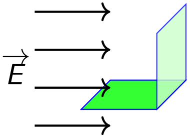


Electric Flux

The electric flux, usually represented by Φ, gives the amount of "flow" of electric field through a surface. In the case where the surface is a plane, if the Electric field and the plane are perpendicular, the expression is given by:

$$\Phi = \left|\vec{E}\right| A \tag{5}$$

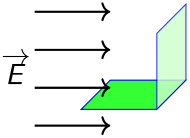
Background - Electric Flux



Q: What is the electric flux through the following surface, where each green sheet has area of 1m^2, and $\left|\vec{E}\right| = 1$ N/C?

1. 0 Nm^2/C
2. 1 Nm^2/C
3. 2 Nm^2/C

Background - Electric Flux



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1. 0 Nm^2/C
2. 1 Nm^2/C
3. 2 Nm^2/C

A: 1 Nm^2/C. There is no flux through the parallel part, and and EA $= 1$ Nm^2/C flux for the perpendicular part.
(Bonus: Can you generalize this for the case where you have a plane that is at an angle θ to the electric field?)

Background - Gauss's Law

Gauss' Law

The flux through a closed surface and the charge enclosed by that closed surface are related by:

$$\Phi = \oint \overrightarrow{E}\, dA = \frac{q_{encl}}{\varepsilon_0} \tag{6}$$

In the case where the electric field is constant along the surface, the above integral reduces to:

$$\Phi = EA = \frac{q_{encl}}{\varepsilon_0} \tag{7}$$

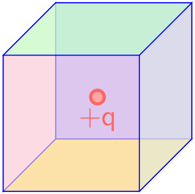
Background - Gauss' Law

Example: Gauss' Law to recover the electric field from a point charge q. We want to choose a surface that fits the symmetry of the problem (i.e. such that the electric field will be constant at all points along the surface, allowing us to use the simplified form of Gauss' Law). What would be the appropriate surface to choose here?

1. A rectangular box
2. A tetrahedron
3. A sphere
4. A cylinder

A: A sphere. The electric field from a point charge is spherically symmetric, so a sphere is a natural choice.

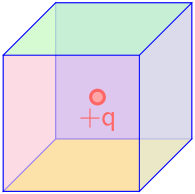
2. Charge[3]



Suppose I have a cube and place a charge $+q$ at the center. What is the flux through one of the faces? (Hint: Recall Gauss' Law that $\Phi = \frac{q_{encl}}{\varepsilon_0}$).

1. 0
2. $\frac{q}{\varepsilon_0}$
3. $\frac{q}{4\varepsilon_0}$
4. $\frac{q}{6\varepsilon_0}$

2. Charge[3]



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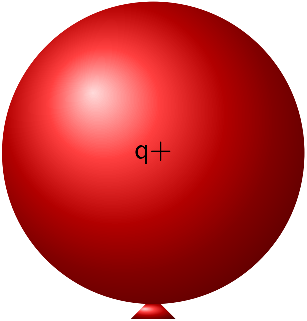
1. 0
2. $\frac{q}{\varepsilon_0}$
3. $\frac{q}{4\varepsilon_0}$
4. $\frac{q}{6\varepsilon_0}$

A: $\frac{q}{6\varepsilon_0}$. Gauss' Law tells us that the total flux through the cube surface is $\frac{q_{encl}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$, and as the flux through each of the faces will be the same by symmetry, the flux through once face is a sixth of the total.

(Bonus: Would this result still hold if the charge was not at the center of the cube?)

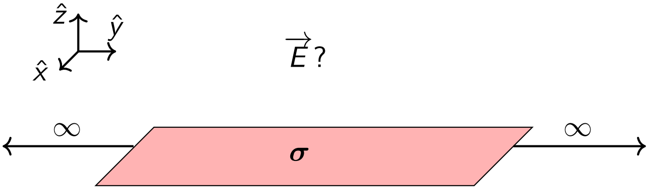
3. A charged balloon



Suppose I have a balloon with charge $+q$ **uniformly distributed** over the surface. What is the electric field inside and outside of the balloon?

3. A charged balloon - Solution

4. An infinite charged plane



What is the electric field from an infinite plane with uniform charge density (charge per unit area) $+\rho$?

4. An infinite charged plane - solution

Background - Electric Potential

Potential

The electric potential V is a scalar function whose rate of change gives the Electric field, i.e. in one dimension:

$$E = -\frac{\Delta V}{\Delta x} \tag{8}$$

The potential of a point charge q is given by:

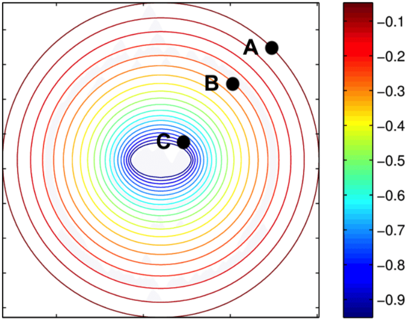
$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \tag{9}$$

Superposition

Like the electric field from multiple sources, the potential from multiple sources $V_1, \cdots, V_n$ is given by:

$$V_{tot} = V_1 + \cdots + V_n \tag{10}$$

Background - Electric Potential

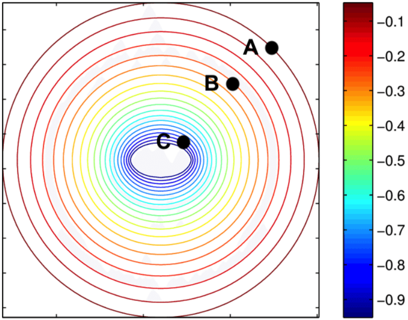


Q: Above is a contour plot of the electric potential for a dust grain in a plasma sheath[1] At which point does the electric field the greatest?

1. Point A
2. Point B
3. Point C

[1] Picture taken from https://www.researchgate.net/publication/33053568

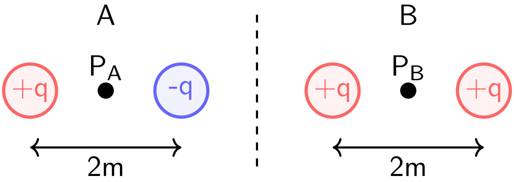
Background - Electric Potential



Q: Above is a contour plot of the electric potential for a dust grain in a plasma sheath[2] At which point does the electric field the greatest?
A: At point C; we recall that the Electric field is given by the rate of change of the electric potential, and as the contour lines are closest at point C, the potential is changing the most at point C and hence $E = -\frac{\Delta V}{\Delta x}$ is maximized there.

[2]Picture taken from https://www.researchgate.net/publication/33053568

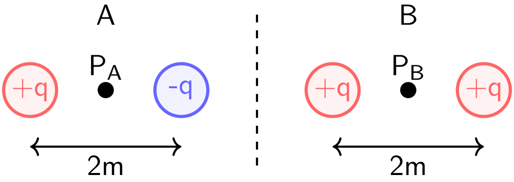
Background - Electric Potential



Q: What is the electric potential in cases A and B at the point P_A, P_B?

1. $\frac{1}{4\pi\varepsilon_0}2q$
2. $\frac{1}{4\pi\varepsilon_0}q$
3. 0
4. $-\frac{1}{4\pi\varepsilon_0}q$
5. $-\frac{1}{4\pi\varepsilon_0}2q$

Background - Electric Potential



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1. $\frac{1}{4\pi\varepsilon_0}2q$
2. $\frac{1}{4\pi\varepsilon_0}q$
3. 0
4. $-\frac{1}{4\pi\varepsilon_0}q$
5. $-\frac{1}{4\pi\varepsilon_0}2q$

A: 0 for case A (the contribution of the positive/negative potential from the $+q/-q$ cancel out) and $\frac{1}{4\pi\varepsilon_0}2q$ for case B (the contribution of the positive potential from the two $+q$s add up).

Background - Induced Charge

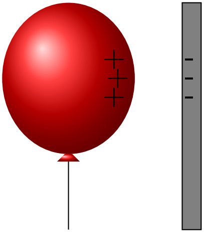
Conductors

In materials that are known as conductors (e.g. metals like copper), the charges are free to move around within the object.

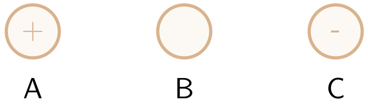
Induced charge

Charged objects can induce a charge on neutral objects, polarizing them

Example: approach a balloon (that you previously rubbed on your hair) to a wall



Background - Induced Charge



Suppose I have three copper balls, one of which has a positive net charge, one of which is electrically neutral, and one of which has negative net charge. Which of the three balls will be attracted to a positively charged rod?

a. Ball A
b. Ball B
c. Ball C

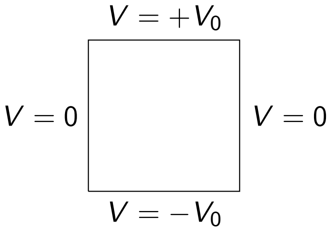
A: Balls B and C. Clearly the negatively charged ball will be attracted to the rod, and since the postive rod induces a negative charge on the closer side of the neutral ball, that gets attracted too!

Background - Boundary Conditions

What is a boundary condition?

A boundary condition is essentially a constraint on a function at the boundary of interest in a problem. In our case, we look at boundary conditions on the potential function V.

Example: A square pipe (cross-section)



Background - Laplace's Equation

Laplace's Equation

The electronic potential satisfies Laplace's equation[a], that is:

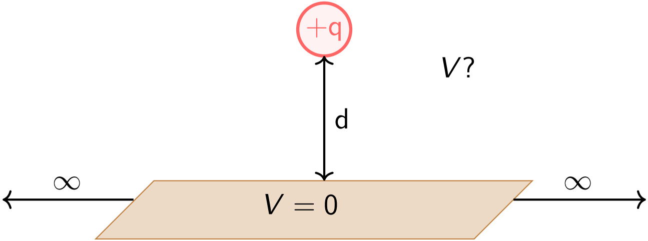
$$\nabla^2 V = 0 \tag{11}$$

[a] In the case where there are no charges in the region of interest

Uniqueness

For a given set of boundary conditions, solutions to Laplace's equation are **unique**; allows us to solve hard problems in a very creative way.

5. Method of Images



Q: We place a point charge $+q$ a distance d above an infinite conducting plane grounded at $V = 0$. What is the potential V in all of space above the plane?

4. Method of Images - solution

Current & Resistance

Current

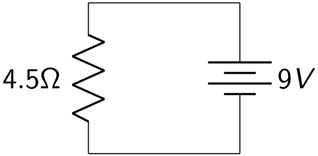
In a circuit, the current I represents rate of the flow of charge with respect to time, i.e. $I = \frac{\Delta q}{\Delta t}$.

Ohm's Law

Given a potential difference V across a material, the current I can be determined from V and the electrical resistance R. The three terms are related by the formula:

$$V = IR \tag{12}$$

Example: Find the current in the given circuit:



Equivalent Resistance

Sometimes we have a complicated resistor configuration, and want to find the equivalent resistance of it.

In series

For resistors in series, the equivalent resistance is given by:

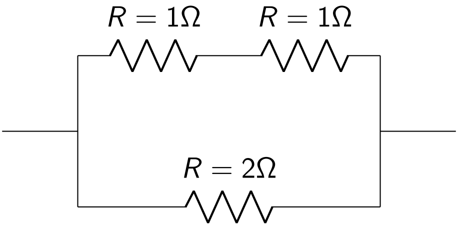
$$R_{eq} = R_1 + R_2 \tag{13}$$

In parallel

For resistors in parallel, the equivalent resistance is given by:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{14}$$

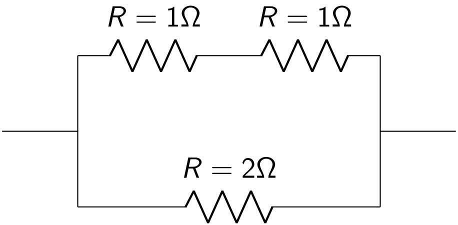
Background - Equivalent Resistance



Q: What is the equivalent resistance of the above three resistors? (Recall that we use $R_{eq} = R_1 + R_2$ for series resistors, $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$ for parallel resistors).

1. 0.5Ω
2. 1Ω
3. 1.5Ω
4. 2Ω

Background - Equivalent Resistance



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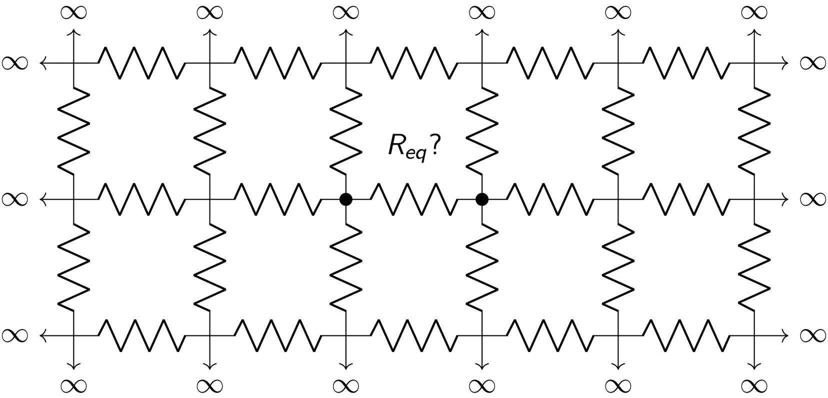
1. 0.5Ω
2. 1Ω
3. 1.5Ω
4. 2Ω

A: 1Ω. Add the two resistors in series to get 2 2Ω resistors in parallel, then add these with the parallel formula to get an total equivalent resistance of 1Ω!

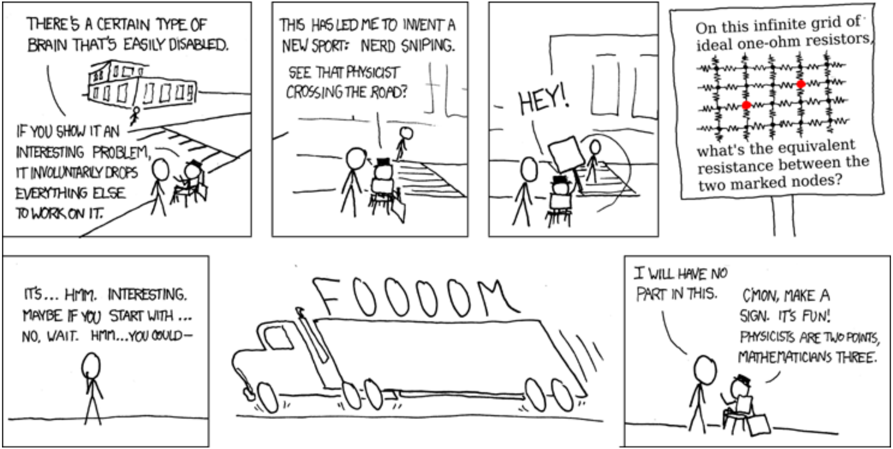
6. Infinite network of resistors



Q: Given the following infinite network of 1Ω resistors, what is the equivalent resistance between two adjacent nodes?

6. Infinite network of resistors - solution

7. Infinite network of resistors - ultra hard mode



Outlook

Symmetry helps us to solve (what appear to be) very hard problems.
Today we looked at Electrostatics only, but there are also applications in...

- Quantum mechanics (Symmetries tell us about energy levels)
- Classical mechanics (Conservation laws)
- Particle physics (representations of quantum states of elementary particles)
- Basically every field....

Moral: Recognizing and exploiting symmetry is a indispensable skill as a physicist!