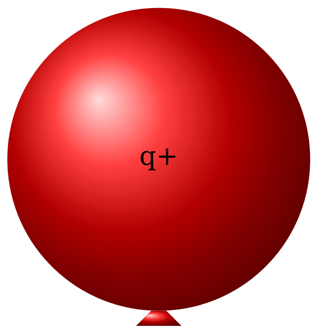
UBC Physics Circle

Symmetry in Electrostatics - Solutions

January 28, 2021

A charged balloon

Or really any spherically symmetric hollow object



In this problem, we will consider the electric field a (perfectly spherical) balloon with total charge $+q$ evenly distributed over its surface. Note that you will need to use Gauss' Law:

$$\oint \vec{E} dA = \frac{q_{encl}}{\epsilon_0}$$

Where in the special case where the electric field has symmetry and is constant across a surface, reduces to:

$$\left|\vec{E}\right| A = \frac{q_{encl}}{\epsilon_0}$$

1. In what way is this situation symmetric? What does this tell you about what the electric field might look like?

   Solution. There is a spherical symmetry, since the balloon is perfectly spherical and there is charge plastered over it uniformly. We would expect the Electric field to be radially symmetric as well. □

2. Imagine I draw an imaginary spherical surface inside of the balloon. What would be the charge enclosed by this surface?

   Solution. If we drew an imaginary spherical surface inside of the balloon, it would contain zero charge (all of the charge is on the surface of the balloon). □

3. Using Gauss's Law, what can you say about the Electric field inside of the balloon?

   Solution. As the situation is spherically symmetric, we can apply Gauss's Law:

   $$\left|\vec{E}\right|A = \frac{q_{encl}}{\epsilon_0}$$

   But the enclosed charge q_{encl} is zero, so we obtain that:

   $$\left|\vec{E}\right| = 0$$

   and hence there is no electric field inside of the balloon. □

4. Now, consider an imaginary spherical surface outside of the balloon (i.e. a larger sphere that encompasses the balloon). What would be the charge enclosed by this surface?

   Solution. The charge enclosed by this surface would be q; just the total amount of charge on the surface of the balloon, since it encompasses the whole thing! □

5. Using Gauss's Law, what can you say about the Electric field outside of the balloon? (Recall that the surface area of a sphere is given by $4\pi r^2$ where r is the radius). Does it remind you of anything else? We have a spherical surface, so by the symmetry of the electric field, the electric field must have the same strength at all points along the surface. Our surface has area $A = 4\pi r^2$ where r is the radius of the sphere (the distance from the origin/the center of the balloon). Applying Gauss's Law, we have:

   $$\left|\vec{E}\right|A = \frac{q_{encl}}{\epsilon_0} \implies \left|\vec{E}\right|4\pi r^2 = \frac{q}{\epsilon_0}$$

   And solving for the electric field, we have:

   $$\left|\vec{E}\right| = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

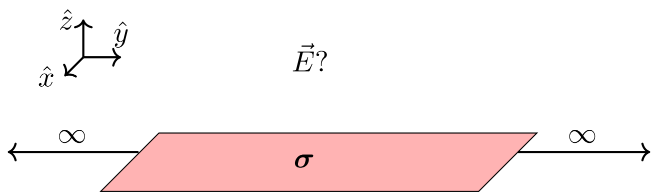
   Since the electric field points radially outwards:

   $$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

   From which we see that the electric field of this balloon is actually identical to that of a point charge with charge q!

An infinite plane of charge

And deriving the energy density of the E-field, if time permits.



In this problem, we will look at solving for the electric field created by an infinite charged sheet in the xy-plane with charge density σ. Note that you will need to use Gauss' Law:

$$\oint \vec{E} dA = \frac{q_{encl}}{\epsilon_0}$$

Where in the special case where the electric field has symmetry and is constant across a surface, reduces to:

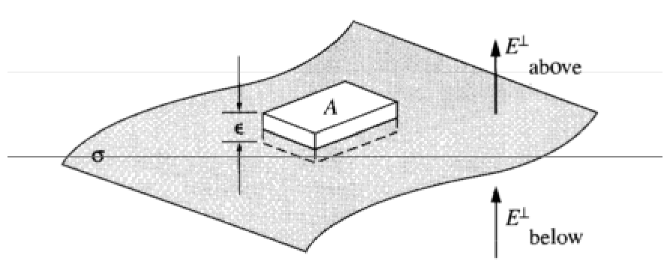
$$\left|\vec{E}\right| A = \frac{q_{encl}}{\epsilon_0}$$

1. Using symmetry, argue why the electric field cannot point in the x or the y direction, and cannot have a dependence on x or y.

   Solution. Since the plane looks the same no matter how much we travel in the x or y direction, by symmetry it wouldn't be able to affect the Electric field. Also by symmetry, every contribution to the x or y direction electric field gets cancelled out by a different charge on the plane, so there can't be an electric field x or y component. □

2. We now must think about a surface that will suit the symmetry of this problem and allow us to apply Gauss's Law. In the case of a point charge for example, since the field is spherically symmetric, the appropriate choice of a surface would be a sphere. What kind of symmetry is present for this problem, and what surface can we choose such that the electric field is constant along the surface?

   Solution. The appropriate symmetry to choose would be a Gaussian pillbox that straddles the surface of the plane. The two parallel (to the plane) surfaces of the pillbox are a distance ϵ away from the plane, and for an infinitely large plane, the Electric field is constant all along these surfaces. We can say these surfaces have area A.



$\square$

3. Determine the charge enclosed by the surface you chose in Q2 above. (Hint: If a sheet has uniform charge density σ and area A, what is the total charge of the sheet?)

   Solution. The total charge enclosed would just be the area of the top face of the pillbox times the charge density, i.e. σA. $\square$

4. Combine everything from Q2 and Q3 to isolate for the electric field in the expression $\left|\vec{E}\right| A = \frac{q_{encl}}{\epsilon_0}$. How does the electric field depend on the distance z from the charged sheet? Is this surprising?

   Solution. We have that the total area that feels an electric field is $2A$ (above and below the plane) so Gauss' Law reads:

$$\left|\vec{E}\right| 2A = \frac{q_{encl}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

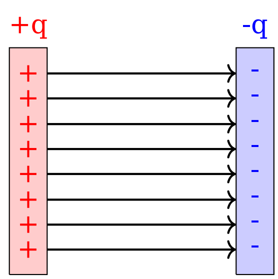
   Isolating for the E-field:

$$\left|\vec{E}\right| = \frac{\sigma}{2\epsilon_0}$$

   We found that this can only point in the z-direction, so:

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{z}$$

   Surprisingly, the Electric field has no distance dependence! The electric field from the plane is the same no matter how far away you are from it (one way of thinking about it is if you zoom out, more charges "come into view"). $\square$



5. (Bonus) The next part of this question is totally optional (though I suppose so is the entirety of physics circle), but will walk through how we can use the result from Q4 to derive the energy density of the electric field. To start, we consider a parallel plate capacitor, which is a circuit component used to store energy. It consists of two parallel plates (one with positive, one with negative charge) with an electric field in between. It turns out that its a very good approximation to treat these parallel plates as infinite (these capacitors might seem small to us, but they certainly "look" infinite on the scale of a proton!). If the plates have charge $+q$ and $-q$ respectively, and have area A, what is the coloumb force from one plate on the other? (Hint: Use the result from Q4 and the relationship $\vec{F} = q\vec{E}$)

Solution. We found that the electric field strength from an infinite plane is given by $\left|\vec{E}\right| = \frac{\sigma}{2\epsilon_0}$. Here the positive plate has surface charge density $\sigma = \frac{q}{A}$ (uniform) so the force from the positive to the negative plate is given by:

$$F = -qE = -q\frac{q}{A2\epsilon_0} = -\frac{q^2}{2A\epsilon_0}$$

I've left off the direction here, but we already know that the force between the two plates is attractive and this would determine the direction. □

6. (Bonus) What is the total electric field strength between the plates?

   Solution. By principle of superposition, we can just add up the contribution of the two plates. Since the electric field that both place produce are in the same direction, we get:

   $$\left|\vec{E}_{total}\right| = \frac{q}{A2\epsilon_0} + \frac{q}{A2\epsilon_0} = \frac{q}{A\epsilon_0}$$

   □

7. (Bonus) Now, suppose the plates start by touching each other, and I separate them a distance d. At the end, what is the volume of space in between the plates (where the electric field lives)?

   Solution. The volume of space between the plates if they are separated by distance d is simply $V = Ad$. □

8. (Bonus) How much work do I have to put in to separate the plates a distance d? (Hint: Use the result from Q5 and the relationship that $W = F\Delta x$ for a constant force)

   Solution. The work done by separating the plates a distance d is:

   $$W = F\Delta x = \frac{q^2}{2A\epsilon_0}d$$

   The force is constant through the whole process, as we already found that the force has no distance dependence for infinite plates. □

9. (Bonus) Combine the results from Q6, Q7 and Q8 to derive the energy density of the electric field in terms of the electric field itself. (*Remark:* Although it might seem like this result only holds for this special case, it is actually true for any electric field!)

   Solution. Dividing the Energy from Q8 by the volume in Q7 (this gives the correct units) we find the energy density u to be:

   $$u = \frac{\text{Energy}}{\text{Volume}} = \frac{\frac{q^2}{2A\epsilon_0}d}{Ad} = \frac{q^2}{2A^2\epsilon_0}$$

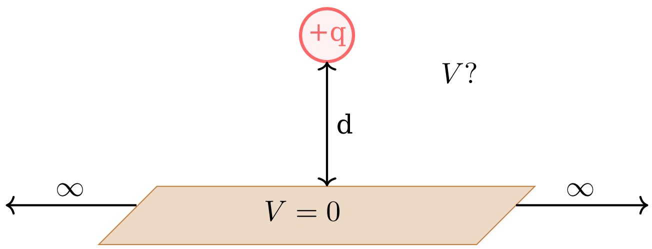
Now, we recognize that $\left|\vec{E}\right| = \frac{q}{A\epsilon_0}$ from Q6, so making this substitution, we have:

$$u = \frac{\epsilon_0}{2}E^2$$

Which is the desired result. It only depends on the electric field strength, and is a totally general formula! ☐

The Method of Images

The coolest way to solve differential equations.



In this problem, we will go through how to solve for the electric potential function (which would in turn allow us to find the Electric field) in all of space when we have a positive charge $+q$ held a distance $z = d$ over a conducting sheet grounded at $V = 0$ in the xy-plane.

1. What are the relevant boundary conditions for this problem?

   Solution. The relevant boundary condition is that $V = 0$ on the xy-plane. ☐

2. Is induced charge relevant to this problem? If so, where can we find it and what is the sign of it? Why does this make solving this problem difficult?

   Solution. Yes, it is relevant; the positive charge will induce some negative charge on the surface of the conducting plane below it. This makes this problem difficult as its hard to know what kind of contribution those negative induced charges are making to the overall potential in space. ☐

3. Mathematicians have (kindly) proved for us that Laplace's equation (that dictates the solutions for V) has unique solutions for given boundary conditions. Hence, if we can think of a simpler scenario that replicates the boundary conditions that we found in Q1, it turns out that the potential function will be the exactly the same for our slightly complicated situation with the grounded conducting plane. Using only point charges, how can we replicate the boundary condition from Q1?

Solution. We can replicate the boundary condition that the plane has a $V = 0$ potential by placing a second "image" charge of the opposite sign (i.e. charge $-q$) a distance d *below* the sheet. □

4. Solve for the potential function V in all of space for the simpler configuration you found in Q3, using the principle of superposition. Recall that the potential for a point charge is given by $V = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|}$ where r_0 is the location of the charge.

   Solution. We have a charge $+q$ at $\vec{r}_+ = (0, 0, d)$ and a charge $-q$ at $\vec{r}_- = (0, 0, -d)$. By superposition the potential in all space is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\vec{r}-\vec{r}_+} + \frac{1}{4\pi\epsilon_0}\frac{-q}{\vec{r}-\vec{r}_-} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\vec{r}-\vec{r}_+} - \frac{1}{\vec{r}-\vec{r}_-}\right)$$

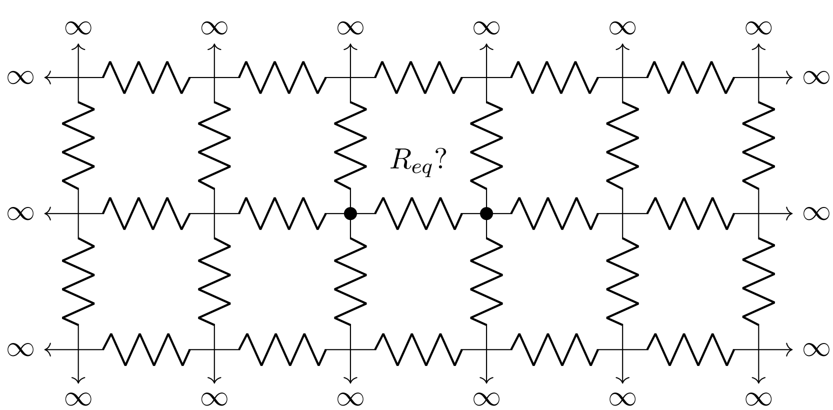
   Using that $|\vec{r}| = \sqrt{x^2+y^2+z^2}$, the potential in terms of the Cartsian coordinates is:

$$V(x, y, z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

□

The infinite grid of resistors

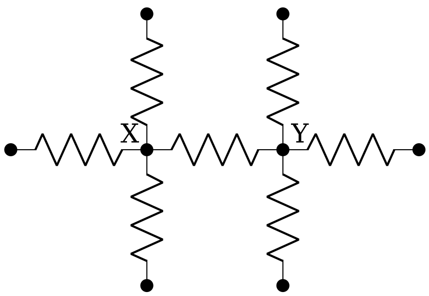
Relevant xkcd. `https://xkcd.com/356/`



In this problem, we will solve for the equivalent resistance between two adjacent nodes (inspired by the famous xkcd comic linked above, though we will be solving an easier problem than pictured there).

1. Start by drawing out a small subsection of the infinite grid, just consisting of two adjacent nodes (label these X and Y), and all of the resistors/nodes adjacent to them (you should have a picture of 8 nodes and 7 resistors in total).

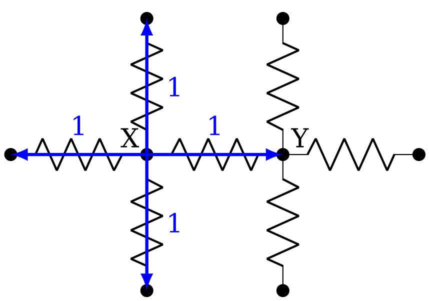
Solution.



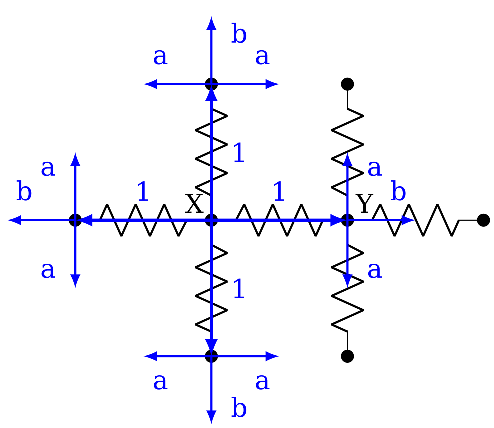
□

2. Suppose we were to inject $I = 4A$ worth of current into node X. Draw out how you expect the current to flow to the adjacent nodes, and to what amounts. Then repeat the same process for how the current flows out of those nodes (use the symmetry of the problem!).

   Solution. Since the lattice is infinite and looks the same in every direction, if we inject 4 Amperes into node X, then an equal amount should flow into the adjacent nodes (1 Ampere each). This is drawn below:



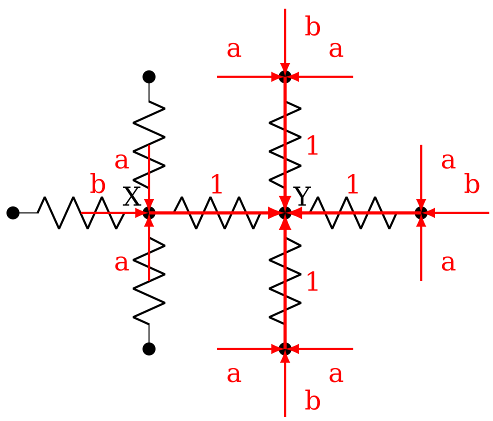
   Then, the 1 Ampere of current will flow out of each of those nodes; some amount a will flow out of two directions (again by symmetry) and some remaining amount $b = 1 - 2a$ will flow out of the remaining direction, as pictured below:



□

3. Next, suppose we were to extract $I = 4A$ worth of current from node Y. Draw out how you expect current to flow from the adjacent nodes to Y, and again repeat the process for how the current flows into those nodes (again use the symmetry of the scenario!)

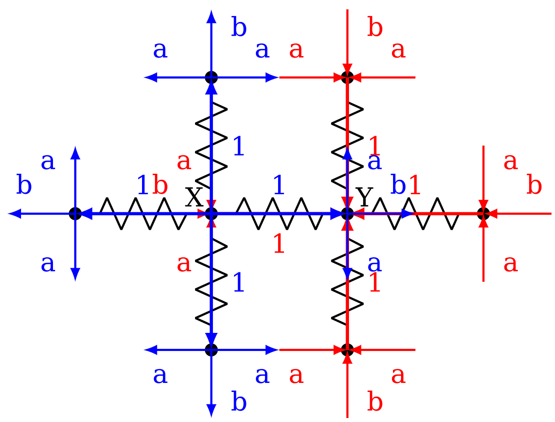
   Solution. The setup is identical to that just asked; just now the sign of the current flips, and our node of interest is Y and not X. Hence we can just flip the sign of the current from the previous part and shift the node from Y to X to obtain the result shown below!



□

4. Now, superimpose the current injection/extraction from Q2 and Q3 onto one diagram. How much current flows from X to Y through the direct connection resistor between them? How much current flows from X to Y through other paths?

   Solution. Superimposing the current injection/extraction patterns, we find:



   From this we can see that 2 Amperes of current flows from X to Y through the resistor directly connecting them, and 2 Amperes of current flows from X to Y through the rest of the resistor network. □

5. If you found that the current was the same between the direct connection and everything else, what does that imply about the resistance of the direct connection between the rest of the grid? If you found that the two were different, what does that imply?

   Solution. Since we found that the current was the same between the direct connection and everything else, this implies that the resistance of the direct connection and the rest of the network are the same; they are both 1 Ohm! □

6. Since the direct connection resistor and the rest of the grid are parallel, use the formula for addition of parallel resistors $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$ to solve for the equivalent resistance between nodes A and B. This is the desired result!

   Solution. Since we found that the direct resistance and the rest of the grid both have resistance $R_{\text{direct}} = R_{\text{rest}} = 1\Omega$, we can use the parallel resistance addition formula to find:

$$\frac{1}{R_{\text{entire}}} = \frac{1}{R_{\text{direct}}} + \frac{1}{R_{\text{rest}}} \implies R_{\text{entire}} = \frac{R_{\text{direct}} R_{\text{rest}}}{R_{\text{direct}} + R_{\text{rest}}} = \frac{1}{2}$$

   so the resistance of the entire infinite resistor network is half an Ohm. □