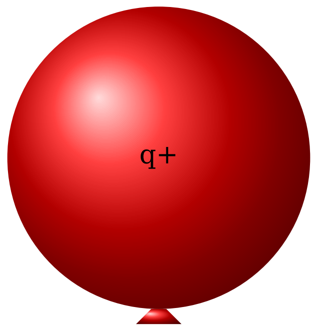
UBC Physics Circle

Symmetry in Electrostatics

January 28, 2021

A charged balloon

Or really any spherically symmetric hollow object



In this problem, we will consider the electric field a (perfectly spherical) balloon with total charge $+q$ evenly distributed over its surface. Note that you will need to use Gauss' Law:

$$\oint \vec{E} dA = \frac{q_{encl}}{\epsilon_0}$$

Where in the special case where the electric field has symmetry and is constant across a surface, reduces to:

$$\left|\vec{E}\right| A = \frac{q_{encl}}{\epsilon_0}$$

1. In what way is this situation symmetric? What does this tell you about what the electric field might look like?

2. Imagine I draw an imaginary spherical surface inside of the balloon. What would be the charge enclosed by this surface?

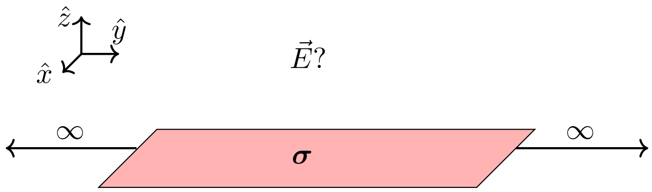
3. Using Gauss's Law, what can you say about the Electric field inside of the balloon?

4. Now, consider an imaginary spherical surface outside of the balloon (i.e. a larger sphere that encompasses the balloon). What would be the charge enclosed by this surface?

5. Using Gauss's Law, what can you say about the Electric field outside of the balloon? (Recall that the surface area of a sphere is given by $4\pi r^2$ where r is the radius). Does it remind you of anything else?

An infinite plane of charge

And deriving the energy density of the E-field, if time permits.



In this problem, we will look at solving for the electric field created by an infinite charged sheet in the xy-plane with charge density σ. Again, you will need to use Gauss' Law:

$$\oint \vec{E} dA = \frac{q_{encl}}{\epsilon_0}$$

Where in the special case where the electric field has symmetry and is constant across a surface, reduces to:

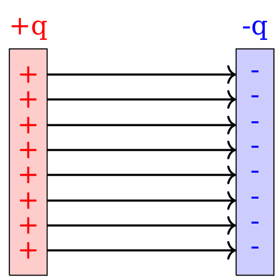
$$\left|\vec{E}\right| A = \frac{q_{encl}}{\epsilon_0}$$

1. Using symmetry, argue why the electric field cannot point in the x or the y direction, and cannot have a dependence on x or y.

2. We now must think about a surface that will suit the symmetry of this problem and allow us to apply Gauss's Law. In the case of a point charge that we just did, since the field is spherically symmetric, the appropriate choice of a surface was a sphere. What kind of symmetry is present for this problem, and what surface can we choose such that the electric field is constant along the surface?

3. Determine the charge enclosed by the surface you chose in Q2 above. (Hint: If a sheet has uniform charge density σ and area A, what is the total charge of the sheet?)

4. Combine everything from Q2 and Q3 to isolate for the electric field in the expression $\left|\vec{E}\right| A = \frac{q_{encl}}{\epsilon_0}$. How does the electric field depend on the distance z from the charged sheet? Is this surprising?



5. (Bonus) The next part of this question is totally optional, but will walk through how we can use the result from Q4 to derive the energy density of the electric field. To start, we consider a parallel plate capacitor, which is a circuit component used to store energy. It consists of two parallel plates (one with positive, one with negative charge) with an electric field in between. It turns out that its a very good approximation to treat these parallel plates as infinite (these capacitors might seem small to us, but they certainly "look" infinite on the scale of a proton!). If the plates have charge $+q$ and $-q$ respectively, and have area A, what is the Coloumb force from one plate on the other? (Hint: Use the result from Q4 and the relationship $\vec{F} = q\vec{E}$)

6. (Bonus) What is the total electric field strength between the plates?

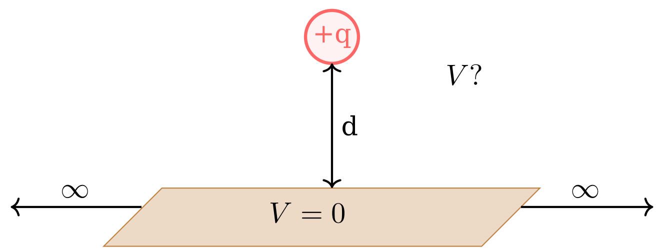
7. (Bonus) Now, suppose the plates start by touching each other, and I separate them a distance d. At the end, what is the volume of space in between the plates (where the electric field lives)?

8. (Bonus) How much work do I have to put in to separate the plates a distance d? (Hint: Use the result from Q5 and the relationship that $W = F\Delta x$ for a constant force)

9. (Bonus) Combine the results from Q6, Q7, and Q8 to derive the energy density of the electric field. (*Remark:* Although it might seem like this result only holds for this special case, it is actually true for any electric field!)

The Method of Images

The coolest way to solve differential equations.



In this problem, we will go through how to solve for the electric potential function (which would in turn allow us to find the Electric field) in all of space when we have a positive charge $+q$ held a distance $z = d$ over a conducting sheet grounded at $V = 0$ in the xy-plane.

1. What are the relevant boundary conditions for this problem?

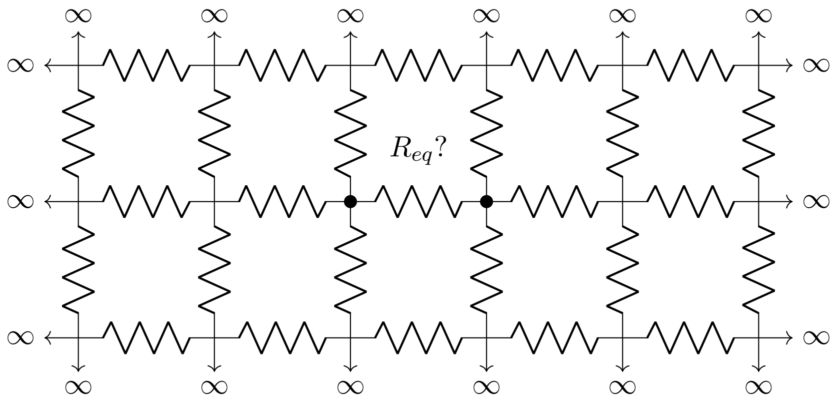
2. Is induced charge relevant to this problem? If so, where can we find it and what is the sign of it? Why does this make solving this problem difficult?

3. Mathematicians have (kindly) proved for us that Laplace's equation (that dictates the solutions for V) has unique solutions for given boundary conditions. Hence, if we can think of a simpler scenario that replicates the boundary conditions that we found in Q1, it turns out that the potential function will be the exactly the same for our slightly complicated situation with the grounded conducting plane. Using only point charges, how can we replicate the boundary condition from Q1?

4. Solve for the potential function V in all of space for the simpler configuration you found in Q3, using the principle of superposition. Recall that the potential for a point charge is given by $V = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|}$ where r_0 is the location of the charge.

The infinite grid of resistors

Relevant xkcd. *`https://xkcd.com/356/`*



In this problem, we will solve for the equivalent resistance between two adjacent nodes (inspired by the famous xkcd comic linked above, though we will be solving an easier problem than pictured there).

1. Start by drawing out a small subsection of the infinite grid, just consisting of two adjacent nodes (label these X and Y), and all of the resistors/nodes adjacent to them (you should have a picture of 8 nodes and 7 resistors in total).

2. Suppose we were to inject $I = 4A$ worth of current into node X. Draw out how you expect the current to flow to the adjacent nodes, and to what amounts. Then repeat the same process for how the current flows out of those nodes (use the symmetry of the problem!).

3. Next, suppose we were to extract $I = 4A$ worth of current from node Y. Draw out how you expect current to flow from the adjacent nodes to Y, and again repeat the process for how the current flows into those nodes (again use the symmetry of the scenario!)

4. Now, superimpose the current injection/extraction from Q2 and Q3 onto one diagram. How much current flows from X to Y through the direct connection resistor between them? How much current flows from X to Y through other paths?

5. If you found that the current was the same between the direct connection and everything else, what does that imply about the resistance of the direct connection between the rest of the grid? If you found that the two were different, what does that imply?

6. Since the direct connection resistor and the rest of the grid are parallel, use the formula for addition of parallel resistors $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$ to solve for the equivalent resistance between nodes A and B. This is the desired result!