PHYS 143 Discussion Week 7 - Diffraction

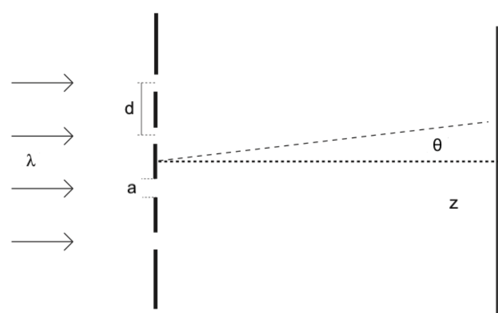
Rio Weil

This document was typeset on May 13, 2025

In this tutorial we go through an exam-style problem involving interference/diffraction, followed by an (outside of the scope of the course) look at how single-slit diffraction can be derived from quantum mechanics (in particular, telling us that quantum mechanical particles - like beams of light - can diffract!)

1 N-slit Interference & Single-slit Diffraction

(Adapted from an exam problem from MIT's 8.03SC) A monochromatic source of plane waves of wavelength λ illuminates a four slit grating, as depicted below. The length of the slits is perpendicular to the page, and the screen is very distant from the slits, with $d \ll z$.



(a) Write an expression in terms of d, λ, θ for the intensity I that will be viewed on the screen. Assume that the slits are very narrow compared to their separation, so $a \ll d$. Assume that the intensity of light due to one slit is I_0.

(b) Make a sketch of the intensity as a function of $kd \sin\theta$ for the four-slit grating, specifying the locations of interference principal maxima and minima.

(c) Now consider the same grating with the two inner slits blocked. Write an expression for the intensity observed on the screen and make the sketch of the new intensity vs. $kd \sin\theta$.

(d) Compare the two sketches - what are the new locations of the maxima and minima? Which principal maxima are at the same location? How has the magnitude of the principal maxima changed? Assume that the individual light intensities of the open slits are the same, I_0 for both cases.

(e) We now want to consider the limit where $a \sim d$ so the width of the slits cannot be neglected. In lecture you saw the derivation of the intensity for a single slit of width a by integrating over the electric fields from the slits, finding that:

$$\frac{I(\theta)}{I(0)} = \left(\frac{\sin(\frac{ka\sin\theta}{2})}{\frac{ka\sin\theta}{2}}\right)^2. \tag{1.1}$$

with $k = \frac{2\pi}{\lambda}$. Let's derive this expression in another way, starting from the N-slit interference intensity for N spacings with slit separation d:

$$I = I_0\left(\frac{\sin(\frac{N\pi d\sin\theta}{\lambda})}{\sin(\frac{\pi d\sin\theta}{\lambda})}\right)^2 = I_0\left(\frac{\sin(\frac{Nkd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})}\right)^2 \tag{1.2}$$

We can consider a single slit of size a as composed of N slits separated by distance d, so $a = Nd$ in the double limit where $d \to 0$ and $N \to \infty$. Suppose also that $I_0(N) = \frac{I_0}{N^2}$, i.e. the intensity of a single slit decreases as $\frac{1}{N^2}$. Show that this reproduces the single-slit diffraction intensity.

(f) Write down the intensity as a function of θ for an N-slit diffraction grating, now taking into account the individual thicknesses of the slits a.

(g) Now, let's return back to the four slit grating with all the slits uncovered. We consider that a is no longer negligible, and in particular $d/a = 2$. The effect of single slit diffraction will cause some of the principle maxima obtained in (a) to disappear. What is the lowest interference order for which diffraction effects zero out the principal maximum in this fashion?

2 Solution

(a) The electric field is obtained for N slits (taking the sum over all electric fields coming from each of the slits, noting that each has a successive path difference of $d \sin\theta$ and hence a phase difference of $kd \sin\theta$):

$$\begin{aligned}
E_{\text{tot}}(\theta) &= \sum_{j=0}^{N-1} E_j \\
&= \tilde{E}_0 \sum_{j=0}^{N-1} e^{ijkd\sin\theta} \\
&= \tilde{E}_0 \frac{1 - e^{ikdN\sin\theta}}{1 - e^{ikd\sin\theta}} \\
&= \tilde{E}_0 \frac{e^{ikdN\sin\theta/2}\left(e^{-ikdN\sin\theta/2} + e^{ikdN\sin\theta/2}\right)}{e^{ikd\sin\theta/2}\left(e^{-ikd\sin\theta/2} + e^{ikd\sin\theta/2}\right)} \\
&= \tilde{E}_0 e^{ikd\sin\theta\frac{N-1}{2}} \frac{\sin(\frac{Nkd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})}
\end{aligned} \tag{2.1}$$

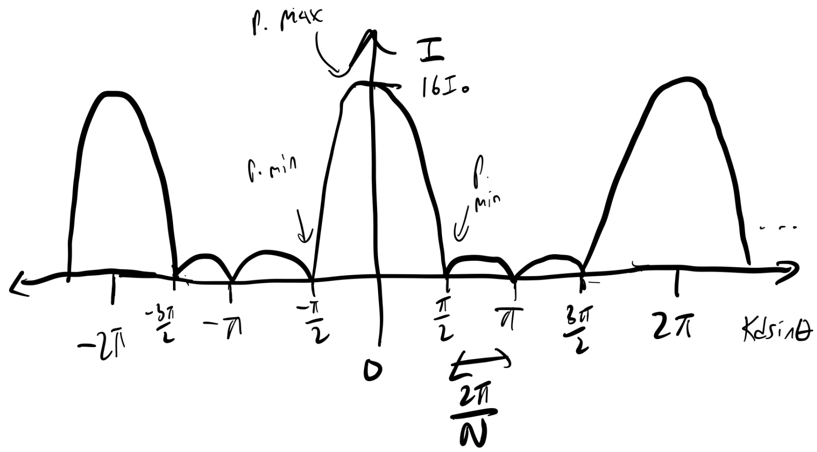
note that $\tilde{E}_0 = \tilde{E}_0(\theta)$ may have an angular dependence, but for small angles we can assume $\tilde{E}_0(\theta) \approx \tilde{E}_0(\theta = 0) = \tilde{E}_0$ a constant. Hence the intensity looks like:

$$I(\theta) = |E_{\text{tot}}(\theta)|^2 = |\tilde{E}_0|^2 \left(\frac{\sin(\frac{Nkd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})}\right)^2 = I_0 \left(\frac{\sin(\frac{Nkd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})}\right)^2 \tag{2.2}$$

where we have used I_0 for the intensity of a single slit, and hence for $N = 4$:

$$\boxed{I(\theta) = I_0 \left(\frac{\sin(\frac{4kd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})}\right)^2} \tag{2.3}$$

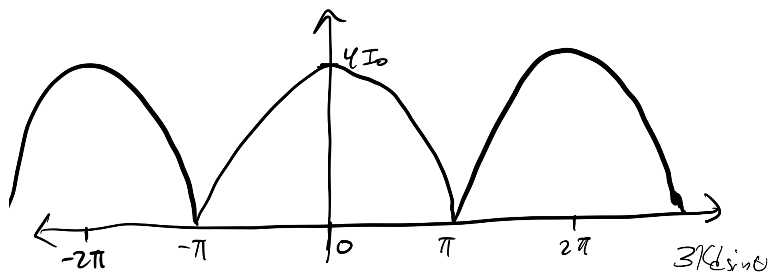
(b) Noting that the maximum intensity is $N^2 I_0$ and that we have spaces of $kd\sin\theta = \frac{2\pi}{N}$ between fringes, we can sketch:

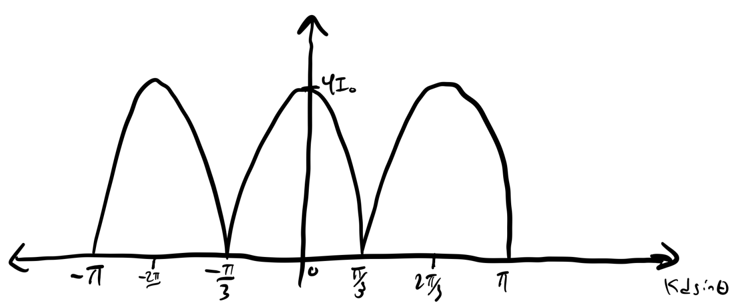


(c) If we block the two inner slits, we now have an $N = 2$ interference pattern with spacing $3d$, so the intensity becomes:

$$\boxed{I(\theta) = I_0 \left(\frac{\sin(\frac{6kd\sin\theta}{2})}{\sin(\frac{3kd\sin\theta}{2})} \right)^2} \tag{2.4}$$

which we can sketch:





(d) The new locations of the maxima are at $kd\sin\theta = \frac{2\pi n}{3}$ and minima at $kd\sin\theta = \frac{2\pi(n+1/2)}{3}$ for $n \in \mathbb{Z}$. The maxima at $2\pi n, n \in \mathbb{Z}$ are at the same location. The magnitude of the principle maxima decreases to $4I_0$.

(e) Taking $Nd = a$, we have:

$$\frac{I(\theta)}{I(0)} = \frac{I_0}{I_0 N^2} \left(\frac{\sin(\frac{ka\sin\theta}{2})}{\sin(\frac{ka\sin\theta}{2N})} \right)^2 = \frac{1}{N^2} \left(\frac{\sin(\frac{ka\sin\theta}{2})}{\sin(\frac{ka\sin\theta}{2N})} \right)^2 \tag{2.5}$$

Taking the $N \to \infty$ limit we can Taylor expand the sin in the denominator to first order:

$$\frac{I(\theta)}{I(0)} = \frac{1}{N^2}\left(\frac{\sin(\frac{ka\sin\theta}{2})}{\frac{ka\sin\theta}{2N}}\right)^2 = \left(\frac{\sin(\frac{ka\sin\theta}{2})}{\frac{ka\sin\theta}{2}}\right)^2 \tag{2.6}$$

which is the single-slit diffraction formula. The correct prefactor is:

$$I(\theta) = |\tilde{E}_0|^2 a^2 \left(\frac{\sin(\frac{ka\sin\theta}{2})}{\frac{ka\sin\theta}{2}}\right)^2 \tag{2.7}$$

(f) To obtain the intensity for N slits, we take the product of the single slit and N-slit interference formulae:

$$\frac{I(\theta)}{I(0)} = \left(\frac{\sin(\frac{ka\sin\theta}{2})}{\frac{ka\sin\theta}{2}}\right)^2 \left(\frac{\sin(\frac{Nkd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})}\right)^2 \tag{2.8}$$

This is a bit heuristic, but we can obtain this more formally by taking the superposition of single-slit integrated electric fields:

$$\begin{aligned} E_{\text{tot}} &= B(0)\left(\int_0^a e^{-iky\sin\theta}dy + \int_d^{d+a} e^{-iky\sin\theta}dy + \ldots + \int_{(N-1)d}^{(N-1)d+a} e^{-iky\sin\theta}dy\right) \\ &= B(0)\underbrace{\left(\int_0^a e^{-iky\sin\theta}dy\right)}_{\text{single-slit}}\underbrace{\sum_{j=0}^{N-1} e^{-ikjd\sin\theta}}_{N-\text{slit}} \end{aligned} \tag{2.9}$$

(g) With $d/a = 2$ then $a = \frac{d}{2}$ so the intensity looks like:

$$\frac{I(\theta)}{I(0)} = \left(\frac{\sin(\frac{kd\sin\theta}{4})}{\frac{kd\sin\theta}{4}}\right)^2 \left(\frac{\sin(\frac{Nkd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})}\right)^2 \tag{2.10}$$

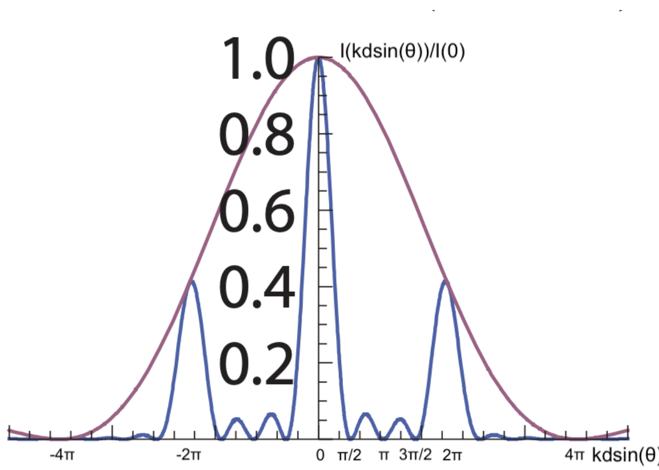
In particular the single-slit diffraction term creates an envelope to the N-slit interference pattern, causing the principle maxima to disappear whenever:

$$\frac{kd\sin\theta}{4} = \pi n, \quad n \neq 0 \tag{2.11}$$

i.e.:

$$kd\sin\theta = 4\pi n, \quad n \neq 0 \tag{2.12}$$

hence the single-slit diffraction effects zero out the principle maxima at $kd\sin\theta = 4\pi n$ for $n \neq 0$. A sketch is given below (adapted from Morin), with the magenta the envelope arising from the single slit diffraction, and the blue the combined pattern.



3 Quantum Single-Slit Diffraction

In quantum mechanics, the state of a particle is described by the wavefunction $\psi(x,t)$, which evolves under the Schrodinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi \tag{3.1}$$

and describes the probability of measuring the particle in a given position interval:

$$P(a \le x \le b) = \int_a^b |\psi(x)|^2 dx \tag{3.2}$$

We now consider a slit of size D, and firing through it a particle of mass m. What is the wavefunction that would describe this? Well, it has uniform probability to be found anywhere inside of the slit, so we can say:

$$\psi(x) = \begin{cases} A & -D/2 \le x \le D/2 \\ 0 & \text{elsewhere} \end{cases} \tag{3.3}$$

i.e. a constant in the slit and zero elsewhere. The normalization condition:

$$1 = P(-\infty < x < \infty) = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \tag{3.4}$$

can be used to find:

$$1 = \int_{-D/2}^{D/2} |A|^2 dx = D|A|^2 \implies A = \frac{1}{\sqrt{D}} \tag{3.5}$$

Now, we think about the momentum probability distribution of the particle. Momentum is defined as the generator of translations:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}, \quad \hat{T}(\Delta x) = \exp(-i\frac{\hat{p}}{\hbar}\Delta x) \tag{3.6}$$

$$(T(\Delta x)\psi)(x) = \psi(x + \Delta x) \tag{3.7}$$

The eigenstates of the momentum operator are those with wavefunctions:

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \tag{3.8}$$

Where we can see that these are eigenstates/under the $\hat{p}$ operator. Thus, the momentum wavefunction can be defined as:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi(x) dx \tag{3.9}$$

i.e. the basis expansion of $\psi(x)$ in fourier/momentum space, wherein:

$$P(k_1 \leq p \leq k_2) = \int_{k_1}^{k_2} |\tilde{\psi}(p)|^2 dp \tag{3.10}$$

Taking the fourier transform of our position wavefunction in the slit:

$$\begin{aligned}
\tilde{\psi}(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x) dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-D/2}^{D/2} e^{-ipx/\hbar} dx \\
&= \frac{1}{\sqrt{2\pi\hbar D}} \frac{\hbar}{-ip} e^{-ipx/\hbar} \Big|_{-D/2}^{D/2} \\
&= i\sqrt{\frac{\hbar}{2\pi D}} \frac{1}{p} (-2i \sin(\frac{pD}{2\hbar})) \\
&= \sqrt{\frac{2\hbar}{\pi D}} \frac{\sin(\frac{pD}{2\hbar})}{p}
\end{aligned} \tag{3.11}$$

So then:

$$|\tilde{\psi}(p)|^2 = \frac{2\hbar}{\pi D} \frac{\sin^2(\frac{pD}{2\hbar})}{p^2} \tag{3.12}$$

This tells us the x-distribution of momenta, which then defines the probability that the particle will hit the detector screen at a certain x-position - the functional form of this is precisely the single-slit diffraction we saw with light classically! There's a bit more work to do here to get the exact intensity pattern $I(\theta)$ on the screen (which you can work out with some trigonometry and), but the analog is clear. This tells us that if we fire quantum-mechanical particles individually through a single slit, we get a pattern - although the figure below is for 2-slit diffraction, you can see how the buildup of individual electron trajectories gives rise to the 2-slit interference pattern of a light beam (figure from Griffiths, Introduction to Quantum Mechanics):