PHYS 143 Discussion Week 6 - Relativistic Doppler Effect and Dispersion

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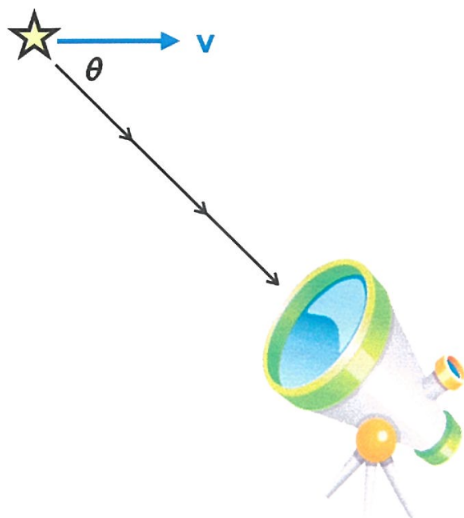
1 Problems

1.1 Relativistic Doppler Effect

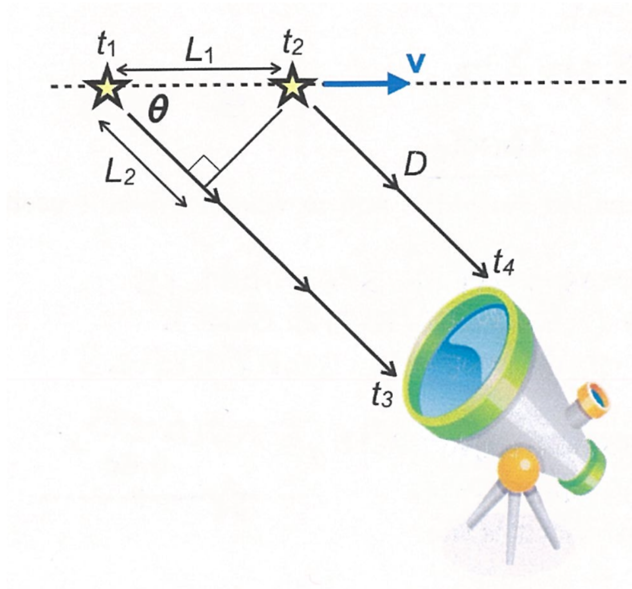
You saw the non-relativistic Doppler effect in class, and somewhat showed the relativistic version, but here we derive it from what we know from relativistic mechanics.



Consider a distance star emitting light with period T in its own frame, moving with $\mathbf{v} = v\hat{\mathbf{x}}$ relative to an observer on Earth. The light has component $v\cos\theta$ towards the observer. We want to calculate the period/frequency that the observer sees. For simplicity, suppose that the pulse that the star emits is as long as one wavelength of light.

(a) What is the time between t_1 (the start of the pulse) and t_2 (the end of the pulse) in the observer frame?

(b) Let $t_1 = 0$. Determine the difference of time between when the beginning of the pulse reaches the telescope at t_3 and when the end reaches the telescope at t_4. Deduce what t_2, t_3, t_4, L_1, L_2 are in the below figure in terms of the given parameters of the question.



(c) Write T_{obs} in terms of T, γ, v, c, θ, and use this to write λ_{obs} and ω_{obs}.

(d) What is λ_{obs} if a star is moving away from the observer? Check also ω_{obs} in this case and that it matches what you saw in class. How could this effect be useful for astronomers?

(e) What is λ_{obs} if the start is moving perpendicular to the observer? Is the shift more or less severe than the case where it is moving away? How does the transverse doppler effect differ in the

(f) (Probably will skip in discussion) It turns out that the redshift:

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{e}}}{\lambda_{\text{e}}} \tag{1.1}$$

can be used to date the age of galaxies in an expanding universe! Let's derive this. If the universe is isotropically expanding (such as ours!), then distances between points have the form $r(t) = a(t)r(t_0)$ where $a(t)$ is the scale factor. In such a universe, the distance that a light beam covers in time dt is given by:

$$cdt = a(t)dr \tag{1.2}$$

By considering a galaxy that emits wavelength λ_{em} and the times of emission/observation, show that the redshift of the galaxy is related to the scale factor as:

$$1 + z = \frac{a(t_o)}{a(t_e)} \tag{1.3}$$

Hence the redshift of a galaxy (or other celestial object) tells us the scale factor at time t_e!

1.2 Dispersion of Exotic Waves

For waves on a string/in air/EM waves, the dispersion relation is linear $\omega = ck$, so we don't see a distinct difference between group and phase velocities. Let's discuss some examples where we do see a distinction!

In quantum mechanics, the state of a particle in space/time is described by the wavefunction $\psi(x,t)$, where the probability of finding the particle in the interval $[a, b]$ is given by:

$$P(a \leq x \leq b) = \int_a^b |\psi(x,t)|^2 dx. \tag{1.4}$$

The evolution of this wavefunction is given by the Schrodinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi \tag{1.5}$$

(a) Consider $V(x) = 0$ (i.e. no potential/the particle is free) and the travelling wave ansatz $\psi(x,t) = Ae^{i(kx-\omega t)}$. Derive the dispersion relation between ω and k. Given that the wave momentum is given by $p = \hbar k$ and the wave energy is given by $E = \hbar\omega$, then what does this dispersion relation remind you of?

(b) Compute the phase velocity and the group velocity. How do they differ? Which one corresponds to the classical velocity of the particle (and why does this make sense)?

There are also cases where the group/phase velocity can differ in some limits, but converge in others. For example, we can derive ocean wave solutions (see https://uw.pressbooks.pub/ocean285/chapter/depth-dependent-ocean-surface-wave-solution/) of the form $\Phi(z,t) = A\cosh[k(z+h)]\cos(kx-\omega t)$ (with h the depth of the water) which have the dispersion relation:

$$\omega^2 = gk\tanh(kh) \tag{1.6}$$

(c) (Probably will skip in discussion) Derive the phase and group velocities in the general case, and check that they differ.

(d) Consider the limit where the water is shallow, and the wavelength is much larger than the depth. What are the phase and group velocies in this case?

(e) Consider the limit where the water is deep, and the wavelength is much smaller than the depth. What are the phase and group velocies in this case?

(f) Imagine a packet of waves in the deep ocean - what happens to the packet in this case as the packet travels? Now, imagine a packet of large-wavelength waves in shallow waters - what happens to the packet in this case as the packet travels? Which one is more dangerous?

2 Solutions

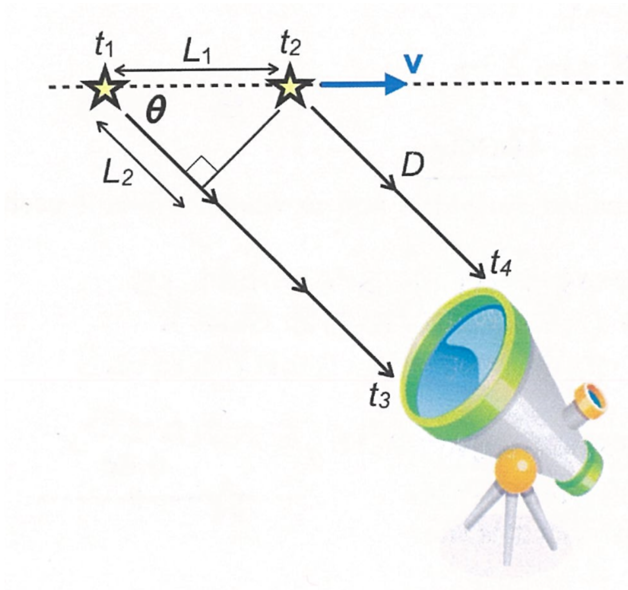
2.1 Relativistic Doppler Effect

(a) Time dilation tells us that:

$$t_2 - t_1 = \gamma T \tag{2.1}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

(b) Referring to the figure



we have that:

$$t_2 = \gamma T \tag{2.2}$$

from the assumption that $t_1 = 0$, and otherwise (from geometry):

$$L_1 = v\gamma T \tag{2.3}$$

$$L_2 = L_1 \cos\theta = \gamma v T \cos\theta \tag{2.4}$$

$$t_3 = \frac{D + L_2}{c} \tag{2.5}$$

$$t_4 = \frac{D}{c} + t_2 \tag{2.6}$$

(c) Computing T_{obs} in terms of what we found in (b):

$$T_{\text{obs}} = t_4 - t_3 = \left(\frac{D}{c} + t_2\right) - \left(\frac{D}{c} + \frac{L_2}{c}\right) = t_2 - \frac{L_2}{c} = \gamma T - \frac{\gamma T v \cos\theta}{c} = \gamma T(1 - \frac{v}{c}\cos\theta) \tag{2.7}$$

Now since $T_{\text{obs}} = \frac{\lambda_{\text{obs}}}{c}$ and $T = \frac{\lambda}{c}$

$$\lambda_{\text{obs}} = \gamma\lambda(1 - \frac{v}{c}\cos\theta) \tag{2.8}$$

We can also find the frequency from $T_{\text{obs}} = \frac{2\pi}{\omega_{\text{obs}}}$ and $T = \frac{2\pi}{\omega}$ so:

$$\omega_{\text{obs}} = \frac{\omega}{\gamma(1 - \frac{v}{c}\cos\theta)} \tag{2.9}$$

(d) If the star is moving away, $\theta = \pi$ and so:

$$\lambda_{\text{obs}} = \gamma(1 + \frac{v}{c})\lambda = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(1 + \frac{v}{c})\lambda = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}\lambda. \tag{2.10}$$

for frequency:

$$\omega_{\text{obs}} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}\omega \tag{2.11}$$

which is what you saw in class. We see that $\lambda_{\text{obs}} > \lambda$, and this can be used by astronomers to study the speed of planets based on shifts to their spectra.

(e) If the star is moving perpendicular, then $\theta = \pi/2$ so:

$$\lambda_{\text{obs}} = \gamma\lambda \tag{2.12}$$

The doppler shift arises from time dilation, vs. in the non-relativistic case arose from sound waves travelling in different speeds at different frames.

(f) Rearranging the given relation:

$$c\,\frac{\mathrm{d}dt}{\mathrm{d}a(t)} = dr \tag{2.13}$$

If the galaxy emits light with wavelength λ_e, and supposing the crest is emitted at time t_e and observed at t_o:

$$c\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_0^r dr = r \tag{2.14}$$

The next crest is emitted at $t_e = \lambda_e/c$ and observed at $t_0 + \lambda_0/c$, so:

$$c\int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c} \frac{dt}{a(t)} = \int_0^r dr = r \tag{2.15}$$

hence:

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c} \frac{dt}{a(t)} \tag{2.16}$$

Which subtracting $\int_{t_e+\lambda_e/c}^{t_o} \frac{dt}{a(t)}$ from both sides:

$$\int_{t_e}^{t_e+\lambda_e/c} \frac{dt}{a(t)} = \int_{t_o}^{t_o+\lambda_o/c} \frac{dt}{a(t)} \tag{2.17}$$

Since $a(t)$ is effectively a constant in the integrals above (assuming the universe expands sufficiently slowly - which is true given our measured data), we can take it out:

$$\frac{1}{a(t_e)}\int_{t_e}^{t_e+\lambda_e/c} dt = \frac{1}{a(t_o)}\int_{t_o}^{t_o+\lambda_o/c} dt \tag{2.18}$$

so:

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_o}{a(t_o)} \tag{2.19}$$

and so:

$$\frac{\lambda_e}{\lambda_o} = 1 + z = \frac{a(t_o)}{a(t_e)} = \frac{1}{a(t_e)} \tag{2.20}$$

where by convention $a(t_o) = 1$.

2.2 Dispersion of Exotic Waves Solution

(a) Putting in the travelling wave ansatz:

$$i\hbar(-i\omega)Ae^{i(kx-\omega t)} = -\frac{\hbar^2}{2m}(ik)^2 Ae^{i(kx-\omega t)} \implies \hbar\omega = \frac{\hbar^2 k^2}{2m} \implies \omega = \frac{\hbar k^2}{2m} \tag{2.21}$$

Given that $p = \hbar k$ and $E = \hbar\omega$, this reminds us of the classical relationship of the (kinetic) energy and momentum of a free particle, $E = \frac{p^2}{2m}$.

(b) The phase velocity is given by:

$$v_p = \frac{\omega}{k} = \frac{\hbar k}{2m} \tag{2.22}$$

The group velocity is given by:

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}\frac{\hbar k^2}{2m} = \frac{\hbar k}{m} \tag{2.23}$$

The classical velocity of the particle is $v = \frac{p}{m}$, so corresponds to the group velocity. This makes sense as the group velocity describes the velocity of a packet/excitation of a wave (and hence corresponds to a particle trajectory), while the phase velocity might describe the motion of a delocalized probability wave (and not a localized particle in space).

(c) The phase velocity is given by:

$$v_p = \frac{\omega}{k} = \frac{\sqrt{gk\tanh(kd)}}{k} \tag{2.24}$$

The group velocity we obtain by implicitly differentiating the expression for ω^2 with respect to k:

$$2\omega\frac{\mathrm{d}\omega}{\mathrm{d}k} = g\tanh(kd) + gkd\,\mathrm{sech}^2(kd) \tag{2.25}$$

Hence:

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{g\tanh(kd) + gkd\,\mathrm{sech}^2(kd)}{2\omega} \tag{2.26}$$

And substituting ω:

$$v_g = \frac{g\tanh(kd) + gkd\,\mathrm{sech}^2(kd)}{2\sqrt{gk\tanh(kd)}} \tag{2.27}$$

(d) In the limit where $kh \ll 1$, then $\tanh(kh) \approx kh$ so:

$$\omega^2 \approx ghk^2 \implies \omega = \sqrt{gh}k \tag{2.28}$$

at this point we already know that the phase/group velocities will be the same since the dispersion is linear. The phase velocity is given by:

$$v_p = \frac{\omega}{k} = \sqrt{gh} \tag{2.29}$$

and the group velocity is obtained via implicit differentiation:

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}\sqrt{gh}k = \sqrt{gh} \tag{2.30}$$

so for shallow water waves $v_p = v_g$.

(e) In the limit where $kh \gg 1$, then $\tanh(kh) \approx 1$ so:

$$\omega^2 = gk \implies \omega = \sqrt{gk} \tag{2.31}$$

the phase velocity is:

$$v_p = \frac{\omega}{k} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}} \tag{2.32}$$

the group velocity is:

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}\sqrt{gk} = \frac{1}{2}\sqrt{\frac{g}{k}} \tag{2.33}$$

so $v_g = \frac{1}{2}v_p$.

(f) From (e) we know a packet of short-wavelength waves in deep water will disperse/travel at different rates, so the waves will disperse in time and not be harmful. Conversely, from (D) a packet of long-wavelength waves in shallow water all stay together and travel at the same rate, so we expect this to be more dangerous.