PHYS 143 Discussion Week 5 - Electromagnetic Waves

Rio Weil

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(Discussion Week 4 was just midterm review, so no write-up).

1 Problem Statement

In this problem, we'll show that a charged particle in the presence of an electromagnetic (monochromatic) plane wave exhibits oscillations in two directions (at different frequencies) plus a drift. Fun fact: I had to solve a version of this problem to pass out of graduate EM1!

(a) Write down the Maxwell equations in the absence of sources (charges and currents).

(b) By taking the curl of Faraday's Law, derive a 3-D wave equation for $\mathbf{E}$.

(c) By using the ansatz of a travelling wave, derive a dispersion relation.

(d) Using Gauss' Law on the above ansatz, show that the direction of propagation $\hat{\mathbf{k}}$ and the direction of the field $\mathbf{E}$ are orthogonal.

(e) Consider a wave of the form:
$$\mathbf{E} = E_0 \cos(kz - \omega t)\hat{\mathbf{x}} \tag{1.1}$$
What direction does the wave travel in (and is this consistent with what you found in (d))? What is the corresponding magnetic field?

(f) Consider now a particle of charge q and mass m. Write down Newton's equation for the particle. Show that if $\mathbf{r}(t=0) = \dot{\mathbf{r}}(t=0) = 0$ (the particle is at rest) then the motion is in the (x, z) plane at all times.

(g) In physics, it is common to treat problems perturbatively - solving an easier/analytically tractable problem first and then adding on more complicated terms as (smaller) corrections to the initial solution. To leading order, set the magnetic field to zero in the equations of (f) and solve for the position of the particle in the (x, z) plane with the given initial conditions.

(h) Now, add the leading effect of the magnetic field on the z-component of the trajectory of the particle, by substituting in the trajectory. You may assume the long wavelength approximation where $kz \ll 1$ (or in terms of the wavelength, $\frac{z}{\lambda} \ll 1$). What is the self consistency condition on q, m, ω, E_0 for the calculation to be reliable?

2 Solution

(a) With $\rho = 0, \mathbf{J} = \mathbf{0}$, the Maxwell equations read:

$$\nabla \cdot \mathbf{E} = 0 \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{2.4}$$

(b) Taking the curl of Eq. (2.2):

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \tag{2.5}$$

where we have used that partial derivatives commute. Now if we use $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ on the LHS and apply Ampere's Law Eq. (2.4) on the RHS, then:

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.6}$$

Now using Gauss' Law Eq. (2.1) on the LHS, the first term vanishes and we are left with:

$$\boxed{\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}} \tag{2.7}$$

which is the desired wave equation.

(c) Substituting in the travelling wave ansatz $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ into the wave equation, we find the dispersion:

$$(i\mathbf{k})^2 \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \frac{1}{c^2}(-i\omega)^2 \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \implies \omega^2 = c^2|\mathbf{k}|^2 \implies \boxed{\omega = c|\mathbf{k}|} \tag{2.8}$$

(d) Applying Gauss' Law:

$$\nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = 0 \implies \boxed{\mathbf{k} \cdot \mathbf{E}_0 = 0} \tag{2.9}$$

so it must be that $\hat{\mathbf{k}}$ and $\hat{\mathbf{E}}$ are orthogonal.

(e) The wave travels in $\boxed{\hat{\mathbf{z}}}$, which is consistent with our result in (d) as this is orthogonal to $\hat{\mathbf{x}}$. To find the $\mathbf{B}$-field, we use Faraday's Law:

$$\nabla \times (E_0 \cos(kz - \omega t)\hat{\mathbf{x}}) = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.10}$$

Evaluating the curl:

$$-E_0 k \sin(kz - \omega t)\hat{\mathbf{y}} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.11}$$

Thus:

$$\mathbf{B}(t) = \int E_0 k \sin(kz - \omega t)\hat{\mathbf{y}} = \frac{E_0 k}{\omega}\cos(kz - \omega t)\hat{\mathbf{y}} = \boxed{\frac{E_0}{c}\cos(kz - \omega t)\hat{\mathbf{y}}} \tag{2.12}$$

(f) Newton's 2nd law combined with hte Lorentz force law reads:

$$m\ddot{\mathbf{r}} = \mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}) \tag{2.13}$$

Which if we evaluate the cross product and write in vector notation reads:

$$\boxed{\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \frac{q}{m} \begin{pmatrix} E_0 \cos(kz - \omega t) - \dot{z}\frac{E_0}{c}\cos(kz - \omega t) \\ 0 \\ \dot{x}\frac{E_0}{c}\cos(kz - \omega t) \end{pmatrix}} \tag{2.14}$$

Since We see that if $y(t=0) = \dot{y}(t=0) = 0$ and there are no forces in $\hat{\mathbf{y}}$, it must be that $\boxed{y(t) = 0}$ for all time and the motion is confined to the (x, y) plane.

(g) Setting the magnetic field terms to zero, we have:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \frac{q}{m} \begin{pmatrix} E_0 \cos(kz - \omega t) \\ 0 \\ 0 \end{pmatrix} \tag{2.15}$$

We then find that $\boxed{z(t) = 0}$ for all time (as $z(t=0) = \dot{z}(t=0) = 0$) at leading order. Solving for the x-motion, we have:

$$\ddot{x} = \frac{qE_0}{m}\cos(-\omega t) = \frac{qE_0}{m}\cos(\omega t) \implies x(t) = -\frac{qE_0}{m\omega^2}\cos(\omega t) + At + B \tag{2.16}$$

Now applying the initial conditions of $x(t=0) = \dot{x}(t=0) = 0$ we get:

$$A = 0, B = \frac{qE_0}{m\omega^2} \tag{2.17}$$

so:

$$\boxed{x(t) = \frac{qE_0}{m\omega^2}\left[1 - \cos(\omega t)\right]} \tag{2.18}$$

(h) We now look at the leading order z-motion, plugging in our leading order $x(t)$ solution. We note that:

$$\dot{x}(t) = \frac{qE_0}{m\omega}\sin(\omega t) \tag{2.19}$$

and so:

$$\ddot{z} = \frac{q}{m}\frac{qE_0}{m\omega}\sin(\omega t)\frac{E_0}{c}\cos(kz - \omega t) \approx \frac{q^2E_0^2}{m^2\omega c}\sin(\omega t)\cos(\omega t) = \frac{q^2E_0^2}{2m^2\omega c}\sin(2\omega t) \tag{2.20}$$

Which has general solution:

$$z(t) = -\frac{q^2E_0^2}{8m^2\omega^3 c}\sin(2\omega t) + Ct + D \tag{2.21}$$

With $z(t=0) = \dot{z}(t=0) = 0$ we find:

$$C = \frac{q^2E_0^2}{4m^2\omega^2 c}, D = 0 \tag{2.22}$$

And so:

$$\boxed{z(t) = -\frac{q^2E_0^2}{8m^2\omega^3 c}\sin(2\omega t) + \frac{q^2E_0^2}{4m^2\omega^2 c}t} \tag{2.23}$$

The self-consistency condition is for the force from the magnetic field to be small in comparison to the electric field, so:

$$\frac{\frac{q}{m}\dot{x}\frac{E_0}{c}\cos(kz-\omega t)}{\frac{qE_0}{m}\cos(kz-\omega t)} = \frac{\dot{x}}{c} \ll 1 \tag{2.24}$$

which is the non-relativistic limit; if we substitute in our expression for x:

$$\frac{\frac{qE_0}{m\omega}\sin(\omega t)}{c} \sim \boxed{\frac{qE_0}{m\omega c} \ll 1} \tag{2.25}$$

Note that we also had our second approximation of the long wavelength approximation $kz \ll 1$; of the two terms for $z(t)$ found above, the linear drift term is more significant (the sine term is bounded in time, and is smaller in magnitude assuming ω is of order 1 or larger), so the consistency condition on this is:

$$k\frac{q^2E_0^2}{4m^2\omega^2 c}t \ll 1 \implies \frac{q^2E_0^2}{4m^2\omega c^2}t = \boxed{\frac{1}{4}\left(\frac{qE_0}{m\omega c}\right)^2 \omega t \ll 1} \tag{2.26}$$

so this is not respected for all times (which makes sense - after a long enough time, the electron will have drifted past the length scale of a wavelength), but since $\frac{qE_0}{m\omega c}$ is small it is obeyed for $\omega t \sim \mathcal{O}(1)$.