PHYS 143 Discussion Week 3 - Fourier Series and Transform

Rio Weil

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Problem 1 - Sawtooth Series

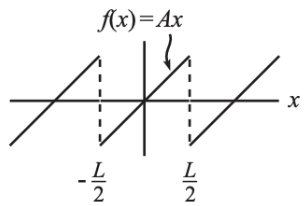

Figure 4

Consider the sawtooth function sketched above, with $f(t) = At$ for $-\frac{T}{2} \leq t \leq \frac{T}{2}$ and $f(t) = f(t+T)$. (We replace $x \to t$, $L \to T$ in the above)

(a) Expand $f(t)$ with trigonometric functions of the same period T:

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(\frac{2\pi nt}{T}) + b_n \sin(\frac{2\pi nt}{T})) \tag{0.1}$$

and determine a_0, a_n, b_n.

(b) Now, expand $f(t)$ using an exponential series of the same period:

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{i\frac{2\pi nt}{T}} \tag{0.2}$$

and determine c_n.

(c) Fourier transform $f(t)$ over the entire range:

$$f(t) = \int \tilde{f}(\omega) e^{i\omega t} d\omega \tag{0.3}$$

and determine $f(\omega)$.

Problem 2 - Basel Problem

You have likely seen the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{0.4}$$

before. Indeed, from convergence tests you also likely know that this converges. But how do we determine its value? Using Fourier series, there is actually a very easy way to compute it!

In particular, Consider the fourier series expansion of $f(x) = x^2$ for $-\pi \leq x \leq \pi$ using sines and cosines. Then, evaluate this expression at $x = \pi$ to find that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{0.5}$$

Bonus Problem (no solution included) - $\sum_n \frac{1}{n^4}$

By considering the Fourier series of the function $f(x) = x^4 - 2\pi^2 x^2$ from $-\pi$ to π, show that:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{0.6}$$

Solution 1

(a) All of the a_n coefficients vanish, because looking at:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} dt At \cos(\frac{2\pi nt}{T}) = 0 \tag{0.7}$$

using that the integrand is an odd function. For the b_ns, we have:

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} dt At \sin(\frac{2\pi nt}{T}) \tag{0.8}$$

Using integration by parts, we know that:

$$\int x \sin(rx) dx = -\frac{x}{r}\cos(rx) + \frac{1}{r^2}\sin(rx) \tag{0.9}$$

and thus:

$$\begin{aligned} b_n &= \frac{2A}{T}\left[-t \left(\frac{T}{2\pi n}\cos(\frac{2\pi nt}{T})\right)\Bigg|_{-T/2}^{T/2} + \left(\frac{T}{2\pi n}\right)^2 \sin(\frac{2\pi nt}{T})\Bigg|_{-T/2}^{T/2} \right] \\ &= \left(-\frac{AT}{2\pi n}\cos(\pi n) + \frac{AT}{2\pi n}\cos(-\pi n)\right) + 0 \\ &= (-1)^{n+1}\frac{AT}{\pi n} \end{aligned} \tag{0.10}$$

Hence we have the Fourier series:

$$f(t) = \frac{AT}{\pi}\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}\sin(\frac{2\pi nt}{T}) \tag{0.11}$$

Which we can plot:

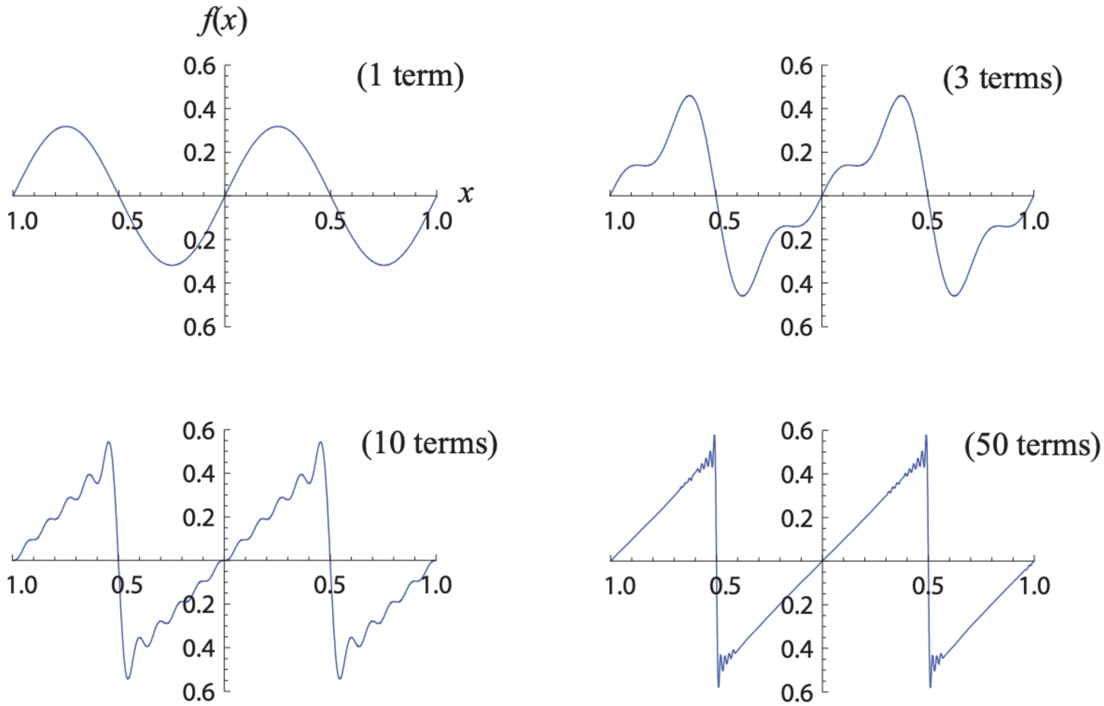

Figure 5

to see that the more terms we add, we can converge towards the form of the sawtooth. Note that there is an overshoot at $\pm\frac{T}{2}$ - this phenomenon, known as the Gibbs phenomenon, is a general phenomena when looking at Fourier series.

Note also that if we take $t = \frac{T}{4}$:

$$f(\frac{T}{4}) = A\frac{T}{4} = \frac{AT}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin(\frac{\pi n}{2}) \implies \frac{\pi}{4} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1} \tag{0.12}$$

so Fourier series give us nice sum identities! We will explore another famous example in the last problem. Note that the above result is also what you get if you evaluate the Taylor series of $\arctan(x)$ at $x = 1$.

(b) Now we compute:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} dt A t e^{-i2\pi n t/T} \tag{0.13}$$

If $n = 0$, then we have the integral of an odd function, so $c_0 = 0$. For $n \neq 0$, we can use the result from integration by parts that:

$$\int x e^{-rx} dx = -\frac{x}{r}e^{-rx} - \frac{1}{r^2}e^{-rx} \tag{0.14}$$

and so:

$$c_n = -\frac{A}{T}\left(\frac{tT}{i2\pi n}e^{-i2\pi nt/T}\Bigg|_{-T/2}^{T/2} + \left(\frac{T}{i2\pi n}\right)^2 e^{-i2\pi nt/T}\Bigg|_{-T/2}^{T/2}\right)$$
$$= -\frac{A}{T}\frac{\left(\frac{T}{2}\right)T}{i2\pi n}(e^{-i\pi n}+e^{i\pi n}) + \frac{A}{T}\left(\frac{T}{i2\pi n}\right)^2(e^{-i\pi n}-e^{i\pi n}) \tag{0.15}$$
$$= (-1)^n\frac{iAT}{2\pi n}$$

So then:

$$f(t) = \sum_{n\neq 0}(-1)^n\frac{iAT}{2\pi n}e^{i2\pi nt/T} \tag{0.16}$$

(c) To get $\tilde{f}(\omega)$ we take the inverse Fourier transform:

$$\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt f(t)e^{-i\omega t}dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\sum_{n\neq 0}(-1)^n\frac{iAT}{2\pi n}e^{i2\pi nt/T}e^{-i\omega t}$$
$$= \frac{iAT}{4\pi^2}\sum_{n\neq 0}(-1)^n\frac{1}{n}\int_{-\infty}^{\infty} dt e^{i2\pi nt/T-i\omega t} \tag{0.17}$$
$$= \frac{iAT}{4\pi^2}\sum_{n\neq 0}(-1)^n\frac{1}{n}2\pi\delta(\omega-\frac{2\pi n}{T})$$
$$= \frac{iAT}{2\pi}\sum_{n\neq 0}(-1)^n\frac{1}{n}2\pi\delta(\omega-\frac{2\pi n}{T})$$

which (as would be expected for a periodic function) is just a series of delta functions as the frequencies we found in (b).

Solution 2

We want to find the coefficients in the expansion:

$$x^2 = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(n_x)\right] \tag{0.18}$$

Note the factor of $\frac{1}{2}$ on the a_0 as we look at the symmetric interval $[-\pi,\pi]$ (you can check that this is necessary to reproduce the correct expression if we just had a constant function). First finding a_0:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx x^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x^2 = \frac{1}{\pi}\left(\frac{1}{3}x^3\Bigg|_{-\pi}^{\pi}\right) = \frac{\pi^2}{3} \tag{0.19}$$

Then finding a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x^2\cos(nx) = \frac{2}{\pi}\int_{0}^{\pi} dx x^2\cos(nx) = (-1)^n\frac{4}{n^2} \tag{0.20}$$

where we use integration by parts. Then computing b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x^2\sin(nx) = 0 \tag{0.21}$$

where we have used that $x^2 \sin(nx)$ is odd. Hence:

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx) \tag{0.22}$$

Taking $x = \pi$ we find $\cos(n\pi) = (-1)^n$ and so the above becomes:

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \tag{0.23}$$

which we can rearrange to get:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}} \tag{0.24}$$