PHYS 143 Discussion Week 1 - Coupled Driven Oscillators

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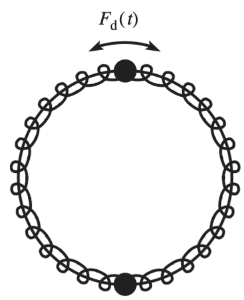
Problem Statement

Consider two identical masses m constrained to move on a horizontal loop, connected by identical springs with spring constant k. Neglect gravity and air resistance. One mass is subject to a driving force $F_d(t) = F_d \cos \omega_d t$. Take $x_1(t), x_2(t)$ to be the coordinates of the top/bottom particle, measured with respect to the top/bottom of the ring, which is the equilibrium position of the springs.



(a) Write down the equations of motion of the system, using Newton's laws.

(b) First, suppose $F_d = 0$. What do you intuitively expect for the normal mode solutions of the system? (Recall the case in class where we "guessed" the center of mass and relative position movement as natural choices for the normal modes).

(c) Derive the homogenous solution to the ODEs you found in (a). Do the normal modes you find match up to what you guessed in (b)?

(d) Next, derive the particular solution, and write down the full solution to the system of ODEs.

(e) What are the equations of motion in the case where:

$$\begin{pmatrix} x_1(t=0) \\ x_2(t=0) \end{pmatrix} = -\frac{F}{\omega_d^2(4\omega^2 - \omega_d^2)} \begin{pmatrix} 2\omega^2 - \omega_d^2 \\ 2\omega^2 \end{pmatrix} \tag{0.1}$$

and:

$$\begin{pmatrix} \dot{x}_1(t=0) \\ \dot{x}_2(t=0) \end{pmatrix} = \begin{pmatrix} v_0 + 2v_1 \\ v_0 - 2v_1 \end{pmatrix} \tag{0.2}$$

Solution

(a) The force on the first particle is $-2k(x_1 - x_2)$ from the two springs and $F_d \cos(\omega_d t)$ from the driving. The force on the second particle is just $-2k(x_2 - x_1)$ from the two springs. Thus Newton's laws read:

$$\begin{aligned} m\ddot{x}_1 &= -2k(x_1 - x_2) + F_d \cos(\omega_d t) \\ m\ddot{x}_2 &= -2k(x_2 - x_1) \end{aligned} \tag{0.3}$$

(b) Intuitively, we might guess there be a mode where the masses move in perfect sync around the ring, without ever compressing/stretching the spring. We might also guess that there is a mode where the masses move equally and opposite towards/away from each other, oscillating in tandem.

(c) We set $F_d = 0$ to find the homogenous solution.

Method 1: Using the Ansatz (note - for *any* homogenous system, we can consider such an ansatz):

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \mathbf{A}e^{\alpha t} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\alpha t} \tag{0.4}$$

The system of equations becomes:

$$m\alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{A}e^{\alpha t} + 2k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{A}e^{\alpha t} = 0 \tag{0.5}$$

Dividing out by m and defining $\omega^2 = \frac{k}{m}$ we have:

$$\begin{pmatrix} \alpha^2 + 2\omega & -2\omega \\ -2\omega & \alpha^2 + 2\omega \end{pmatrix} \mathbf{A}e^{\alpha t} = 0 \tag{0.6}$$

We solve the eigenvalue problem:

$$\det \begin{pmatrix} \alpha^2 + 2\omega & -2\omega \\ -2\omega & \alpha^2 + 2\omega \end{pmatrix} = 0 \implies (\alpha^2 + 2\omega)(\alpha^2 + 2\omega) - (-2\omega)(-2\omega) = \alpha^4 + 4\omega\alpha^2 = \alpha^2(\alpha^2 + 4\omega) = 0 \tag{0.7}$$

which has solutions:

$$\alpha^2 = 0, -4\omega \implies \alpha = 0, \pm 2i\omega \tag{0.8}$$

So we look for eigenvectors:

$$\begin{pmatrix} 0 + 2\omega & -2\omega \\ -2\omega & 0 + 2\omega \end{pmatrix} \mathbf{A} = 0 \implies \begin{pmatrix} 2\omega & -2\omega \\ -2\omega & 2\omega \end{pmatrix} \mathbf{A} = 0 \implies \mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.9}$$

$$\begin{pmatrix} -4\omega + 2\omega & -2\omega \\ -2\omega & -4\omega + 2\omega \end{pmatrix} \mathbf{A} = 0 \implies \begin{pmatrix} -2\omega & -2\omega \\ -2\omega & -2\omega \end{pmatrix} \mathbf{A} = 0 \implies \mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.10}$$

So we have the homogenous solution:

$$\boxed{\mathbf{x}_h(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (C_1 t + C_2) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (C_3 e^{2i\omega t} + C_4 e^{-2i\omega t})} \tag{0.11}$$

There was one step at the end we skipped over - for the multiplicty 2 solution of $\alpha^2 = 0$, how do we get a linear solution $C_1 e^{0t} t$ and a linear solution $C_2 e^{0t}$? First, it is clear that just having $C_1 e^{0t} + C_2 e^{0t} = C_1 + C_2$ as the constant solution is not linearly independent with itself. So, the need for another solution is clear. We can derive it by considering the $\omega \to 0$ limit of the oscillatory solutions $C_1 \sin(\omega t) + C_2 \cos(\omega t)$. Taking the limit of the cosine:

$$\lim_{\omega \to 0} C_2 \cos(\omega t) = C_2 \tag{0.12}$$

we get the constant solution as desired. The sine limit is more subtle, because the limit goes to zero, which is not a new linearly independent solution. But we are free to multiply solutions by a constant, so in particular let's consider $C_2 \sin(\omega t)/\omega$. In this case, the limit becomes:

$$\lim_{\omega \to 0} C_1 \frac{\sin(\omega t)}{\omega} = \lim_{\omega \to 0} C_1 \frac{(\omega t)}{\omega} = C_1 t \tag{0.13}$$

so we recover the linear solution. We'll also see how the linear + constant solution comes out in the second method of getting the homogenous solution, as we discuss below.

Method 2: We can add and subtract the two equations

$$\begin{aligned} \ddot{x}_1 + \ddot{x}_2 &= 0 \\ m(\ddot{x}_1 - \ddot{x}_2) + 4k(x_1 - x_2) &= 0 \end{aligned} \tag{0.14}$$

Which motivates the two normal coordinate definitions $\xi_1(t) = x_1(t) + x_2(t)$ and $\xi_2(t) = x_1(t) - x_2(t)$:

$$\ddot{\xi}_1(t) = 0 \implies \xi_1(t) = C_1 t + C_2 \tag{0.15}$$

$$\ddot{\xi}_2(t) = -4\omega\xi_2(t) \implies \xi_2(t) = C_3 e^{2i\omega t} + C_4 e^{-2i\omega t} \tag{0.16}$$

Which writing in vector form:

$$\boxed{\mathbf{x}_h(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (C_1 t + C_2) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (C_3 e^{2i\omega t} + C_4 e^{-2i\omega t})} \tag{0.17}$$

In both cases, we can see that our intuitive guess for the two normal modes were in fact correct!

(d) We look for a particular solution to:

$$\begin{aligned} m\ddot{x}_1 + 2k(x_1 - x_2) &= F_d \cos(\omega_d t) \\ m\ddot{x}_2 + 2k(x_2 - x_1) &= 0 \end{aligned} \tag{0.18}$$

We could write the RHS of the first equation as $\Re(F_d e^{i\omega_d t})$ and then propose the ansatz:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} e^{i\omega_d t} \tag{0.19}$$

but since the LHS only has even time derivatives, we can make a simpler guess of the form:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \cos(\omega_d t) \tag{0.20}$$

which we can substitute in to obtain (with $\omega^2 = \frac{k}{m}$ and $F = \frac{F_d}{m}$):

$$\begin{aligned} -\omega_d^2 B_1 + 2\omega^2(B_1 - B_2) &= F \\ -\omega_d^2 B_2 + 2\omega^2(B_2 - B_1) &= 0 \end{aligned} \tag{0.21}$$

Adding the two equations:

$$B_1 + B_2 = -\frac{F}{\omega_d^2} \implies B_1 = -\frac{F}{\omega_d^2} - B_2 \tag{0.22}$$

Substituting this into the second equation:

$$-\omega_d^2 B_2 + 2\omega^2 (B_2 + \frac{F}{\omega_d^2} + B_2) = 0 \implies B_2 = -\frac{2F\omega^2}{\omega_d^2(4\omega^2 - \omega_d^2)} \tag{0.23}$$

and then we can also solve for B_1:

$$B_1 = \frac{-F(2\omega^2 - \omega_d^2)}{\omega_d^2(4\omega^2 - \omega_d^2)} \tag{0.24}$$

So the particular solution is:

$$\boxed{\mathbf{x}_p(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\frac{F}{\omega_d^2(4\omega^2 - \omega_d^2)} \begin{pmatrix} 2\omega^2 - \omega_d^2 \\ 2\omega^2 \end{pmatrix} \cos(\omega_d t)} \tag{0.25}$$

The general solution is then the sum of the homogenous and particular solution:

$$\boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \mathbf{x}_p(t) + \mathbf{x}_h(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (C_1 t + C_2) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (C_3 e^{2i\omega t} + C_4 e^{-2i\omega t}) - \frac{F}{\omega_d^2(4\omega^2 - \omega_d^2)} \begin{pmatrix} 2\omega^2 - \omega_d^2 \\ 2\omega^2 \end{pmatrix} \cos(\omega_d t)} \tag{0.26}$$

(e) With the initial condition on the particle positions:

$$-\frac{F}{\omega_d^2(4\omega^2 - \omega_d^2)} \begin{pmatrix} 2\omega^2 - \omega_d^2 \\ 2\omega^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} C_2 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (C_3 + C_4) - \frac{F}{\omega_d^2(4\omega^2 - \omega_d^2)} \begin{pmatrix} 2\omega^2 - \omega_d^2 \\ 2\omega^2 \end{pmatrix} \tag{0.27}$$

Which gives:

$$\begin{aligned} C_2 + C_3 + C_4 &= 0 \\ C_2 - C_3 - C_4 &= 0 \end{aligned} \tag{0.28}$$

Adding these two together we find $C_2 = 0$ and that leaves us with the condition that $C_3 + C_4 = 0$. Next, applying the initial condition of $\dot{\mathbf{x}}(t = 0) = 0$:

$$\begin{pmatrix} v_0 + 2v_1 \\ v_0 - 2v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} C_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (2i\omega C_3 - 2i\omega C_4) \tag{0.29}$$

Which gives:

$$\begin{aligned} C_1 + 2i\omega(C_3 - C_4) &= v_0 + 2v_1 \\ C_1 - 2i\omega(C_3 - C_4) &= v_0 - 2v_1 \end{aligned} \tag{0.30}$$

Adding the two equations we obtain:

$$2C_1 = 2v_0 \implies C_1 = v_0 \tag{0.31}$$

Subtracting the two equations we have:

$$4i\omega(C_3 - C_4) = 4v_1 \implies C_3 - C_4 = -i\frac{v_1}{\omega} \tag{0.32}$$

And combining this with $C_3 + C_4 = 0$ we find:

$$C_3 = -i\frac{v_1}{2\omega}, C_4 = i\frac{v_1}{2\omega} \tag{0.33}$$

Then noting that:

$$-i\frac{v_1}{2\omega}e^{2i\omega t} + i\frac{v_1}{2\omega}e^{-2i\omega t} = \frac{v_1}{\omega}\sin(2\omega t) \tag{0.34}$$

And thus our solution is:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_0 t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{v_1}{\omega}\sin(2\omega t) - \frac{F}{\omega_d^2(4\omega^2 - \omega_d^2)} \begin{pmatrix} 2\omega^2 - \omega_d^2 \\ 2\omega^2 \end{pmatrix} \cos(\omega_d t) \tag{0.35}$$