PHYS 141 Discussion Week 8 - Rotational Motion 2

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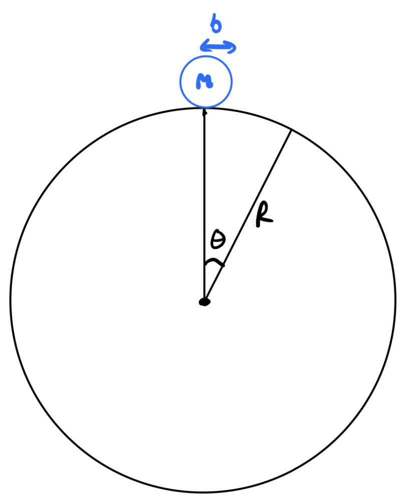
1 Problems

1.1 Marble leaving the sphere (inspired by Morin 8.2)



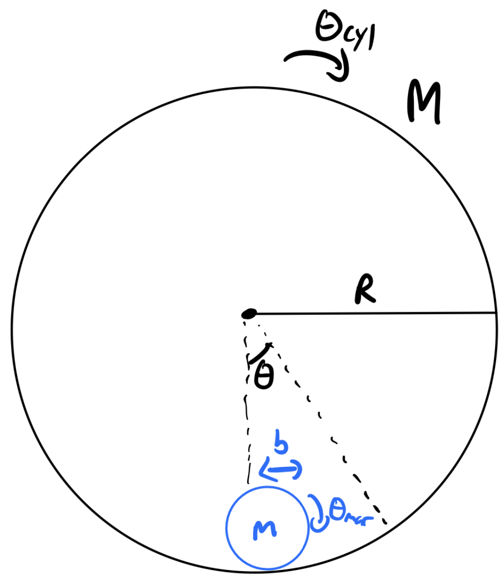
Consider a (spherical) marble of mass m (and uniform mass density) and radius b sitting on top of a sphere of radius $R \gg b$. It is given a small kick, before it rolls down without slipping and then leaves the surface. At what angle θ_c does the sphere leave the sphere?

(a) Write down the radial component of Newton's second law. What forces are acting on the sphere? What corresponds to the marble losing contact with the sphere?

(b) What is conserved in this problem? How can we use this to find the velocity at the time when the marble leaves the sphere?

(c) From this, deduce the angle.

1.2 Marble inside rolling cylinder (inspired by Morin 8.14)



In your homework, you studied a spherical marble of mass m (and uniform mass density), radius b, rolling around without slipping in a circular basin of radius $R \gg b$, and found the frequency of small oscillations of the marble to be:

$$\omega = \sqrt{\frac{5}{7}\frac{g}{R}} \tag{1.1}$$

We now consider a slightly more complicated setup, where the marble is rolling around without slipping in a hollow cylinder of radius $R \gg b$ and mass M, which is also free to roll around. What is now the frequency of small oscillations?

(a) Write down the torque equations/rotational analogs of Newtons' second law, for both the angles θ_1, θ_2 of the marble and the cylinder.

(b) Write down the tangential component of Newton's second law for the marble.

(c) What does the non-slipping condition say in this case?

(d) Combine the results from above to obtain the equation of motion.

(e) Find the small-angle oscillation freuqency

(f) What happens in the limit where $m \ll M$? Does this make sense?

(g) What happens in the limit where $m \gg M$? Does this make sense?

2 Solutions

2.1 Marble leaving the sphere

(a) The radial component of Newton's second law reads:

$$\sum_i F_i^{\text{rad}} = m(\ddot{r} - r\dot{\theta}^2) \tag{2.1}$$

Since we have circular motion while the marble stays on the sphere, we have $r = R$ and $\dot{\theta} = \frac{v}{R}$ and so:

$$\sum_i F_i^{\text{rad}} = \frac{mv^2}{R} \tag{2.2}$$

The forces acting on the marble are gravity and the normal force, so:

$$mg\cos\theta - N = \frac{mv^2}{R} \tag{2.3}$$

The marble leaves the sphere when the normal force vanishes

$$\boxed{g\cos\theta_c = \frac{v^2}{R}} \tag{2.4}$$

(b) Energy is conserved. When the marble leaves the sphere, we have the potential has been converted into kinetic (translational and rotational), so:

$$\Delta U = mgR(\cos(\frac{\pi}{2}) - \cos(\theta_c)) = mgR(1 - \cos\theta_c) = \Delta K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{2.5}$$

Let's study the rotational term. $I = \frac{2}{5}mb^2$ for the spherical marble, and $b = \frac{v}{\omega}$ if it rolls without slipping, so:

$$mgR(1 - \cos\theta_c) = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{2}{5}m\left(\frac{v}{\omega}\right)^2)\omega^2 = \frac{7}{10}mv^2 \tag{2.6}$$

Isolating for v^2:

$$\boxed{v^2 = \frac{10}{7}gR(1 - \cos\theta_c)} \tag{2.7}$$

(c) Substituting our (b) result into our result from (a):

$$g\cos\theta_c = \frac{\frac{10}{7}gR(1 - \cos\theta_c)}{R} \tag{2.8}$$

and isolating for $\cos\theta_c$:

$$\cos\theta_c = \frac{7}{17} \tag{2.9}$$

or:

$$\boxed{\theta_c = \arccos(\frac{7}{17})} \tag{2.10}$$

Surprisingly, this has no dependence on the mass m, g, or the radii b/R; it only depends on the geometry of the problem!

2.2 Marble inside rolling cylinder

(a) The torque equation is given by:

$$\tau = I\alpha \tag{2.11}$$

for both the marble and the cylinder. Let $\theta_{\text{cyl}}, \theta_{\text{mar}}$ be their angles of rotation. The cylinder has moment of inertia $I_{\text{cyl}} = MR^2$ and feels a torque $\tau = -fR$ due to the friction between the cylinder and marble, so:

$$\boxed{-fR = MR^2\ddot{\theta}_{\text{cyl}}} \tag{2.12}$$

The marble has moment of inertia $I_{\text{mar}} = \frac{2}{5}mb^2$ and feels a torque $\tau = fb$ due to the friction, so:

$$\boxed{fb = \frac{2}{5}mb^2\ddot{\theta}_{\text{mar}}} \tag{2.13}$$

(b) Newton's second law for the marble reads:

$$f - mg\sin\theta = mr\ddot{\theta} \tag{2.14}$$

where θ is the angle measured from the center of the well.

(c) The non-slipping condition says:

$$R\theta = R\theta_{\text{cyl}} - b\theta_{\text{mar}} \tag{2.15}$$

(d) We take the second time derivative of both sides of the non-slipping condition:

$$R\ddot{\theta} = R\ddot{\theta}_{\text{cyl}} - b\ddot{\theta}_{\text{mar}} \tag{2.16}$$

If we rearrange our two torque equations from part (a), we have:

$$-\frac{f}{M} = R\ddot{\theta}_{\text{cyl}} \tag{2.17}$$

$$\frac{f}{\frac{2}{5}m} = b\ddot{\theta}_{\text{mar}} \tag{2.18}$$

If we subtract the second equation from the first:

$$-f\left(\frac{1}{M} + \frac{1}{\frac{2}{5}m}\right) = R\ddot{\theta}_{\text{cyl}} - b\ddot{\theta}_{\text{mar}} \tag{2.19}$$

which if we substitute in the expression obtained from the non-slipping condition:

$$f\left(\frac{1}{M} + \frac{1}{\frac{2}{5}m}\right) = -R\ddot{\theta} \implies f = -\frac{1}{\left(\frac{1}{M} + \frac{1}{\frac{2}{5}m}\right)}R\ddot{\theta} \tag{2.20}$$

Substituting this into our expression from (b):

$$-\frac{1}{\left(\frac{1}{M} + \frac{1}{\frac{2}{5}m}\right)}R\ddot{\theta} - mg\sin\theta = mR\ddot{\theta} \tag{2.21}$$

Then:

$$\left(m + \frac{1}{\left(\frac{1}{M} + \frac{1}{\frac{2}{5}m}\right)}\right)\ddot{\theta} + \frac{mg}{R}\sin\theta = 0 \tag{2.22}$$

which after simplifying:

$$\boxed{\left(\frac{\frac{2}{5}m + \frac{7}{5}M}{\frac{2}{5}m + M}\right)\ddot{\theta} + \frac{g}{R}\sin\theta = 0} \tag{2.23}$$

(e) In the limit of small angles $\sin\theta \approx \theta$ and so:

$$\left(\frac{\frac{2}{5}m + \frac{7}{5}M}{\frac{2}{5}m + M}\right)\ddot{\theta} + \frac{g}{R}\theta = 0 \tag{2.24}$$

which corresponds to simple harmonic motion $\theta(t) = A\cos(\omega t + \phi)$ with frequency:

$$\boxed{\omega = \sqrt{\frac{g}{R}\frac{\frac{2}{5}m + M}{\frac{2}{5}m + \frac{7}{5}M}}} \tag{2.25}$$

(f) In the limit $m \ll M$, ω reduces to:

$$\boxed{\omega = \sqrt{\frac{5}{7}\frac{g}{R}}} \tag{2.26}$$

This makes sense as this is the limit where the cylinder doesn't move, i.e. we recover the result from the homework!

(g) In the limit where $m \gg M$, ω reduces to:

$$\boxed{\omega = \sqrt{\frac{g}{R}}}. \tag{2.27}$$

In this limit, the friction force between the two objects goes to zero, so In this limit, the marble acts like a pendulum of length R.