PHYS 141 Discussion Week 7 - Rotational Motion

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1 Problems

1.1 Space Package Delivery (Adapted from K&K 7.4)

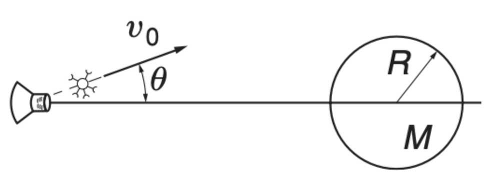

Figure 1.1: Diagram of space package delivery.

A spaceship is sent to investigate a planet of mass M and radius R. While hanging motionless in space at a distance $5R$ from the center of the planet, the ship fires a package with speed v_0, as shown above. The package has mass m which is much smaller than the mass of the spaceship. You as the space-Amazon delivery person must figure out the angle θ to launch the package at such that the package just grazes the surface of the planet.

(a) What quantities are conserved, and why?

(b) Use (a) to show that $\theta = \arcsin(\frac{1}{5}\sqrt{1 + \frac{8}{5}\frac{GM}{Rv_0^2}})$.

(c) Explain why the delivery fails if v_0 is too small, and find the minimum v_0 for which it succeeds.

1.2 Rotating Planet

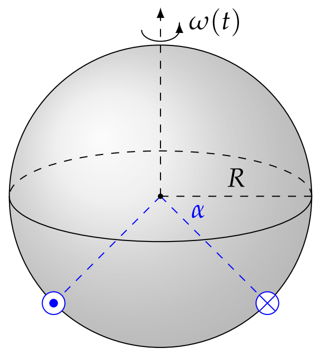

Figure 1.2: Illustration of the planet and the attached rockets (in blue). The left rocket points out of the page, and the right rocket points into the page. Both are located at an angle α from the equator, and induce an angular velocity $\omega(t)$ onto the planet in the direction shown.

Far into the future, humanity is looking to colonize other planets after Earth has become hostile to human life. A new candidate planet was found, but unfortunately it has angular velocity of $\omega_0 = 0$, leading to the stellar-facing side to be too hot and the other to be too cold. To rectify this, NASA engineers decide to strap two high-powered rockets at angles α from the equator, pointing in opposite directions (as shown in the figure - one pointing in one pointing out of the page). From prior measurements taken of the planet, it is known to have mass M and radius R, and uniform mass density. The rockets impart a time-dependent force $F(t) = kt$ (their output ramps up in time). Your goal is to find what time (from when the rockets turn on) the planet completes one full rotation.

(a) Show that the moment of inertia of a uniform-density disk of mass m and radius l about an axis going through the center is given by $I = \frac{1}{2}ml^2$.

(b) Using (a), show that the moment of inertia of the planet rotating about an axis through its center is given by $I = \frac{2}{5}MR^2$.

(c) What is the torque $\boldsymbol{\tau} = \sum_{i=1}^{2} \boldsymbol{\tau}_i = \sum_{i=1}^{2} \mathbf{r}_i \times \mathbf{F}_i$ that the two rockets impart on the planet?

(d) Find the angular velocity $\omega(t)$ of the planet.

(e) Finally, find the time T it takes for the planet to undergo one full rotation.

2 Solutions

2.1 Q1 Solution

(a) $\boxed{\text{Angular momentum about the center of the planet}}$ is conserved, since $\boxed{\text{the gravitational force is central}}$ (and hence applies no torque). Additionally, $\boxed{\text{total mechanical energy}}$ is conserved, since the $\boxed{\text{gravitational force is conservative}}$.

(b) Denote by v the tangential velocity of the package when it grazes the planet. Conservation of angular momentum gives:

$$L_i = L_f \implies m|(\mathbf{v}_i \times \mathbf{r}_i)| = m|(\mathbf{v}_f \times \mathbf{r}_f)| \implies mv_0 5R \sin\theta = mvR. \tag{2.1}$$

where we note that $\mathbf{v}_f$ and $\mathbf{r}_f$ are prependicular. Thus we obtain the relation:

$$5v_0 \sin\theta = v \tag{2.2}$$

Next, conservation of energy gives:

$$K_i + U_i = K_f + U_f \implies \frac{1}{2}mv_0^2 - \frac{GmM}{5R} = \frac{1}{2}mv^2 - \frac{GmM}{R} \implies v^2 = v_0^2 + \frac{8}{5}\frac{GM}{R} \tag{2.3}$$

Substituting in the equation we have from conservation of angular momentum to eliminate v:

$$(5v_0 \sin\theta)^2 = v_0^2 + \frac{8}{5}\frac{GM}{R} \implies \sin\theta = \frac{1}{5}\sqrt{1 + \frac{8}{5}\frac{GM}{Rv_0^2}} \tag{2.4}$$

From which we obtain:

$$\boxed{\theta = \arcsin(\frac{1}{5}\sqrt{1 + \frac{8}{5}\frac{GM}{Rv_0^2}})} \tag{2.5}$$

(c) If v_0 is too small, the RHS of Eq. (2.4) becomes large - since $|\sin\theta| \le 1$, this implies that for small v_0 that the grazing delivery fails (and the package misses the planet). This gives us the constraint that:

$$\frac{1}{5}\sqrt{1 + \frac{8}{5}\frac{GM}{Rv_0^2}} \le 1 \implies \boxed{v_0 \ge \sqrt{\frac{15R}{GM}}} \tag{2.6}$$

2.2 Q2 Solution

(a) The disk has uniform surface mass density $\sigma = \frac{m}{\pi l^2}$. Looking at the moment of inertia, we have:

$$I = \int r^2 dm = \int r^2 \sigma dA = \sigma \int r^2 dA \tag{2.7}$$

now in polar coordinates $dA = rdrd\theta$, so:

$$I = \sigma \int_0^2 \pi d\theta \int_0^l r^2 r dr = \sigma(2\pi)\frac{l^4}{4} \tag{2.8}$$

and substituting in our surface density:

$$\boxed{I = \frac{1}{2}ml^2} \tag{2.9}$$

(b) The mass density of the planet is given by:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} \tag{2.10}$$

We slice up the planet into infinitesimally think disks of mass $dm = \rho dV$ and infinitesimal volume $dV = \pi l^2 dz$. The infinitesimal moment of inertia of one such disk (using (a)) is:

$$dI = \frac{1}{2}dml^2 = \frac{1}{2}\rho\pi l^2 dzl^2 = \frac{1}{2}\rho\pi l^4 dz \tag{2.11}$$

By integrating over the height z, we obtain the moment of inertia of the solid sphere by stacking such infinitesimal disks:

$$I = \int dI = \int_{-R}^{R} \frac{1}{2}\rho\pi l^4 dz \tag{2.12}$$

By pythagoras, $l^2 + z^2 = R^2$, so $l^4 = (R^2 - z^2)^2$ and so:

$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 dz = \rho\pi \int_{0}^{R} (R^2 - z^2)^2 dz \tag{2.13}$$

where the last equality follows via the symmetry of the integrand. The integral is then:

$$I = \rho\pi \int_0^R (R^4 - 2R^2z^2 + z^4)dz = \rho\pi \left(R^5 - \frac{2R^5}{3} + \frac{R^5}{5} \right) = \rho\pi \frac{8R^5}{15} \tag{2.14}$$

then substituting in ρ we find:

$$\boxed{I = \frac{2}{5}MR^2} \tag{2.15}$$

(c) For the left rocket we have $\mathbf{r}_1 = -R\cos\alpha\hat{\mathbf{y}} - R\sin\alpha\hat{\mathbf{z}}$ and $\mathbf{F}_1 = kt\hat{\mathbf{x}}$ and for the right rocket we have $\mathbf{r}_2 = R\cos\alpha\hat{\mathbf{y}} - R\sin\alpha\hat{\mathbf{z}}$ and $\mathbf{F}_2 = -kt\hat{\mathbf{x}}$, which yields a total torque:

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 \\ &= (-R\cos\alpha\hat{\mathbf{y}} - R\sin\alpha\hat{\mathbf{z}}) \times (kt\hat{\mathbf{x}}) + (R\cos\alpha\hat{\mathbf{y}} - R\sin\alpha\hat{\mathbf{z}}) \times (-kt\hat{\mathbf{x}}) \\ &= -2R\cos\alpha kt(\hat{\mathbf{y}} \times \hat{\mathbf{x}}) \end{aligned} \tag{2.16}$$

Therefore:

$$\boxed{\boldsymbol{\tau} = 2R\cos\alpha kt\hat{\mathbf{z}}} \tag{2.17}$$

(d) Since the moment of inertia I of the planet does not change with time (assuming we can neglect the mass of the rockets and how they would change in time as fuel was burned - this would certainly complicate the question!), we can relate the torque τ, the moment of inertia, and the angular velocity via:

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = I\frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{2.18}$$

Thus using our previously known results for the moment of inertia and torque, we obtain the differential equation:

$$\boxed{\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{5k\cos\alpha}{MR}t} \tag{2.19}$$

This can be solved via separation of variables:

$$\int_{\omega(0)}^{\omega(t)} d\omega = \int_0^t \frac{5k\cos\alpha}{MR}dt' \implies \omega(t) = \frac{5k\cos\alpha}{2MR}t^2 + \omega(0) \tag{2.20}$$

Since the planet is not spinning at $t = 0$, we conclude:

$$\boxed{\omega(t) = \frac{5k\cos\alpha}{2MR}t^2} \tag{2.21}$$

(e) Since $\omega = \dfrac{\mathrm{d}\varphi}{\mathrm{d}t}$ where φ is the rotation angle of the planet, we have the differential equation:

$$\boxed{\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{5k\cos\alpha}{2MR}t^2} \tag{2.22}$$

This again we solve via separation of variables:

$$\int_{\varphi(0)}^{\varphi(t)} d\varphi = \int_0^t \frac{5k\cos\alpha}{2MR}t'^2 dt' \implies \phi(t) = \frac{5k\cos\alpha}{6MR}t^3 + \varphi(0) \tag{2.23}$$

Setting $\varphi = 0$ at $t = 0$, we have:

$$\boxed{\varphi(t) = \frac{5k\cos\alpha}{6MR}t^3} \tag{2.24}$$

We are interested in the time T in which $\varphi = 2\pi$, so:

$$\varphi(t = T) = 2\pi = \frac{5k\cos\alpha}{6MR}T^3 \tag{2.25}$$

and isolating for T:

$$\boxed{T = \sqrt[3]{\frac{12\pi MR}{5k\cos\alpha}}} \tag{2.26}$$