PHYS 142 Discussion Week 7 - Magnetic Fields

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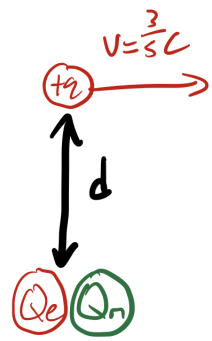
This document was typeset on February 24, 2025

1 Force from Monopoles

A spaceship (modelled as a point) carrying charge $+q$ and with velocity $\mathbf{v} = \frac{3}{5}c\hat{\mathbf{x}}$ passes by/above a pointlike planet consisting of electric + magnetic monopole (located at the origin), with $\mathbf{E}$ and $\mathbf{B}$ fields:

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{Q_e}{r^2}\hat{\mathbf{r}} \\ \mathbf{B} &= \frac{\mu_0}{4\pi}\frac{Q_m}{r^2}\hat{\mathbf{r}} \end{aligned} \tag{1.1}$$

Supposing that the closest point of approach is at $\mathbf{d} = d\hat{\mathbf{y}}$ - at this point, what is the force that we observe acting on the charge in the rest frame? What is the force we observe acting on the charge in the charge/moving frame? (Also: why is this question unphysical, beyond the super-fast travelling spaceship?)



Solution. In the rest frame, at $\mathbf{d} = d\hat{\mathbf{y}}$ the electric/magnetic fields look like:

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{Q_e}{d^2}\hat{\mathbf{y}} \\ \mathbf{B} &= \frac{\mu_0}{4\pi}\frac{Q_m}{d^2}\hat{\mathbf{y}} \end{aligned} \tag{1.2}$$

So using the Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q(\frac{1}{4\pi\epsilon_0}\frac{Q_e}{d^2}\hat{\mathbf{y}} + \frac{3}{5}c\hat{\mathbf{x}} \times \frac{\mu_0}{4\pi}\frac{Q_m}{d^2}\hat{\mathbf{y}}) = \frac{q}{4\pi d^2}\left(\frac{Q_e}{\epsilon_0}\hat{\mathbf{y}} + \frac{3c\mu_0 Q_m}{5}\hat{\mathbf{z}}\right) \tag{1.3}$$

For the moving/charge frame, we first calculate γ:

$$\gamma = \frac{1}{\sqrt{1 - \frac{\left(\frac{3}{5}c\right)^2}{c^2}}} = \frac{5}{4} \tag{1.4}$$

Then we find the fields in the transformed frame to be (noting that all fields here are perpendicular to the direction of motion):

$$\mathbf{E}' = \gamma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = \frac{5}{4}\frac{1}{4\pi d^2}\left(\frac{Q_e}{\epsilon_0}\hat{\mathbf{y}} + \frac{3c\mu_0 Q_m}{5}\hat{\mathbf{z}}\right) \tag{1.5}$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}) = \frac{5}{4}\left(\frac{\mu_0}{4\pi}\frac{Q_m}{d^2}\hat{\mathbf{y}} - \frac{1}{c^2}\frac{3}{5}c\hat{\mathbf{x}} \times \frac{1}{4\pi\epsilon_0}\frac{Q_e}{d^2}\hat{\mathbf{y}}\right) = \frac{5}{4}\frac{q}{4\pi d^2}\left(\mu_0 Q_m\hat{\mathbf{y}} - \frac{3Q_e}{5c\epsilon_0}\hat{\mathbf{z}}\right) \tag{1.6}$$

In the moving frame, the charge has $\mathbf{v}' = 0$ so:

$$\mathbf{F}' = q(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}') = q\mathbf{E}' = \frac{5}{4}\frac{q}{4\pi d^2}\left(\frac{Q_e}{\epsilon_0}\hat{\mathbf{y}} + \frac{3c\mu_0 Q_m}{5}\hat{\mathbf{z}}\right) = \frac{5}{4}\mathbf{F} \tag{1.7}$$

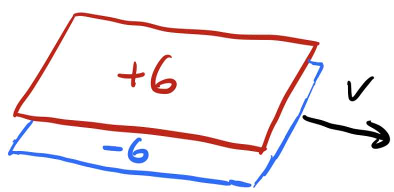
This is just $\gamma\mathbf{F}$, as we might have expected. The situation is unphysical because:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \mu_0 Q_m \delta(\mathbf{r}) \neq 0 \tag{1.8}$$

which violates the Maxwell equations. But theorists like to assume magnetic monopoles may exist out there, somewhere...

2 Capacitor Frisbee

Consider a parallel plate capacitor travelling with velocity $\mathbf{v} = v\hat{\mathbf{x}}$, with surface charge density (as measured in the lab frame) σ.



(a) What is the electric and magnetic fields inside/outside of the plates?

(b) What is the force per unit area on one plate of the capacitor?

(c) What is the speed v at which the capacitor must travel if the forces are to balance?

(d) What do the fields/force look like in the capacitor frame (this should require very little calculation) and what happens in the limit of (c)?

Solution.

(a) The electric field is just the superposition of two infinite sheets of uniform charge density σ, i.e. we will have:

$$\mathbf{E} = \begin{cases} -\frac{\sigma}{\epsilon_0}\hat{\mathbf{z}} & \text{between plates} \\ 0 & \text{elsewhere} \end{cases} \tag{2.1}$$

To find the magnetic field, we first consider just one of the two plates (say, embed it in the xy plane). There is clearly no x-component to the $\mathbf{B}$ field as $\mathbf{B}$ and $\mathbf{v}$ are perpendicular. Further there is no z-component (we can observe that any contribution from a filament of current at $+y$ is cancelled by one at $-y$). So there is only a y component. Now using Ampere's law with an Amperian loop of length l (figure taken from Griffiths - depicted is a sheet of uniform current, which is analogous to if we have a moving sheet of charge):

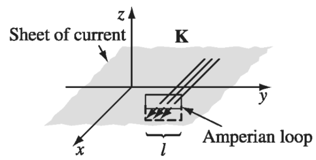


FIGURE 5.33

through which we have enclosed current $I_{\text{encl}} = Kl = \sigma v l = \sigma l v$. We then read off:

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2Bl = \mu_0 I_{\text{encl}} = \mu_0 \sigma l v \tag{2.2}$$

Hence for an infinite moving sheet of charge (i.e. an infinite plane of current):

$$\mathbf{B} = \begin{cases} \frac{\mu_0}{2}\sigma v \hat{\mathbf{y}} & \text{below plate} \\ -\frac{\mu_0}{2}\sigma v \hat{\mathbf{y}} & \text{above plate} \end{cases} \tag{2.3}$$

and superimposing the two distance-independent magnetic fields from the two sheets, we obtain:

$$\mathbf{B} = \begin{cases} \mu_0 \sigma v \hat{\mathbf{y}} & \text{between plates} \\ 0 & \text{elsewhere} \end{cases} \tag{2.4}$$

(b) The attractive electric force per unit area on the upper plate is:

$$\mathbf{F}_e = \sigma \mathbf{E} = -\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{z}} \tag{2.5}$$

The total force for a surface current is $\mathbf{F} = \int \mathbf{K} \times \mathbf{B} d\mathbf{a}$, so the force per unit area is then:

$$\mathbf{K} \times \mathbf{B} \tag{2.6}$$

so with $\mathbf{K} = \sigma v \hat{\mathbf{x}}$ for the upper plate and the lower plate creating a magnetic field $\mathbf{B} = \frac{\mu_0}{2}\sigma v \hat{\mathbf{y}}$, we find:

$$\mathbf{F}_b = \frac{\mu_0}{2}\sigma^2 v^2 \hat{\mathbf{z}} \tag{2.7}$$

so the total force per unit area is:

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_b = \frac{\sigma^2}{2}\left(-\frac{1}{\epsilon_0} + \mu_0 v^2\right)\hat{\mathbf{z}} \tag{2.8}$$

(c) We search for the speed at which the above force vanishes:

$$-\frac{1}{\epsilon_0} + \mu_0 v^2 = 0 \implies v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c \tag{2.9}$$

i.e. the speed of light!

(d) In the capacitor frame, the top plate sees itself and the bottom plate as fixed, so there are no currents/moving charges; hence $\mathbf{B} = \mathbf{0}$. As for the $\mathbf{E}$-field; the lab frame measures a surface density of σ, but in capacitor frame the surface density is reduced to $\sigma_0 = \frac{\sigma}{\gamma(v)}$ (with $\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$), and the electric field is suppressed by the appropriate γ factor. The limit in (c) corresponds to $\lim_{v\to c}\gamma(v) \to \infty$ and hence the fields (and force) vanishes because the surface density in the capacitor frame goes to zero.

3 Vector potential for solenoid

To give us a bit of practice with the vector potential, let us calculate it for an infinite solenoid of radius R with current I and n turns per unit length running through it. From Ampere's law (as you saw in class!) we know that the field inside is uniformly $\mathbf{B} = \mu n I \hat{\mathbf{z}}$ and $\mathbf{B} = \mathbf{0}$ outside.

First, we can use symmetry to constrain the form of $\mathbf{A}$. If we look at the definition of $\mathbf{A}$ in terms of the current:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')dV'}{|\mathbf{r} - \mathbf{r}'|} \tag{3.1}$$

we see that the contributions to $\mathbf{A}$ are all aligned in the direction of the current. Since $\mathbf{J} \propto \hat{\boldsymbol{\theta}}$ everywhere, it must follow that $\mathbf{A} \propto \hat{\boldsymbol{\theta}}$ as well (i.e. the $\hat{\mathbf{r}}, \hat{\mathbf{z}}$ components vanish). Since the system is translation invariant in z (the solenoid is infinite) and rotationally invariant in θ, $\mathbf{A} = A_\theta \hat{\boldsymbol{\theta}}$ can further only depend on r.

Now, we note via Stokes' theorem that the surface integral for the flux Φ through a surface S is equivalent to a path/loop integral on the boundary of S:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{a} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{l} \tag{3.2}$$

Thus we can use the known results of the magnetic field to find the vector potential; for $r < R$/inside the solenoid we have:

$$\Phi = \oint_C \mathbf{A} \cdot d\mathbf{l} \implies B(\pi r^2) = A_\theta(2\pi r) \implies A_\theta = \frac{(\mu_0 n I)(\pi r^2)}{2\pi r} = \frac{\mu_0 n I r}{2} \tag{3.3}$$

and outside the solenoid (for $r > R$) we have:

$$\Phi = \oint_C \mathbf{A} \cdot d\mathbf{l} \implies B(\pi R^2) = A_\theta(2\pi r) \implies A_\theta = \frac{(\mu_0 n I)(\pi R^2)}{2\pi r} = \frac{\mu_0 n I R^2}{2r} \tag{3.4}$$

So:

$$\mathbf{A} = \begin{cases} \frac{\mu_0 n I r}{2} \hat{\boldsymbol{\theta}} & r < R \\ \frac{\mu_0 n I R^2}{2r} \hat{\boldsymbol{\theta}} & r > R \end{cases} \tag{3.5}$$

it can be verified that $\boldsymbol{\nabla} \times \mathbf{A}$ (check it in cylindrical coordinates!) recovers the correct expressions for $\mathbf{B}$ inside/outside the solenoid.

Although completely outside the scope of this course - one argument for why $\mathbf{A}$ is the more physical quantity (rather than the $\mathbf{B}$ field itself) is the fact that $\oint_C \mathbf{A} \cdot d\mathbf{l}$ can be measured via an experiment that measures a quantum-mechanical phase shift arising from particles travelling on the outside of a solenoid. The experiment is able to pick up on the fact that $\oint_C \mathbf{A} \cdot d\mathbf{l} \neq 0$ - even though $\mathbf{B} = \mathbf{0}$ everywhere outside! This is the celebrated *Aharanov-Bohm effect*, and is one motivation for a potential-centric development of electromagnetism (the other motivation being that $\mathbf{A}$ is necessary for when we think about how to couple electromagnetic fields to charge matter, such as in quantum electrodynamics).