PHYS 142 Discussion Week 5 - Resistors and Circuits

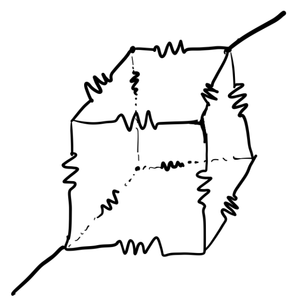
Rio Weil

This document was typeset on February 10, 2025

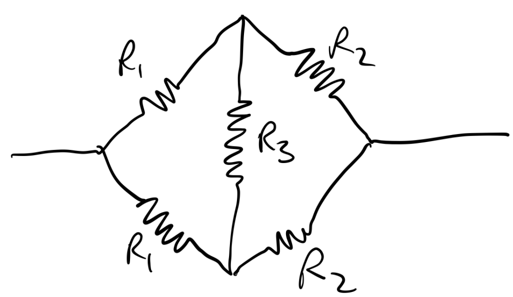
In this tutorial we look at three quick circuit/resistor puzzles, each of which will give us a new trick/method for circuits!

1 Resistor Cube and Equipotentials

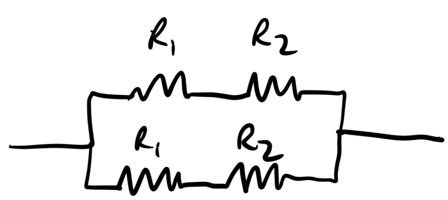
Consider the following cube of resistors, all of equal resistance R:



What is the equivalent resistance of this configuration? First, an easier problem; What is the equivalent resistance of:



To solve this, we note that points A, B are equipotentials. Hence no current flows through the central R_3 resistor. We could also see the fact that no net current flows through the R_3 resistor via symmetry. Either way, we can remove it, giving the equivalent circuit:



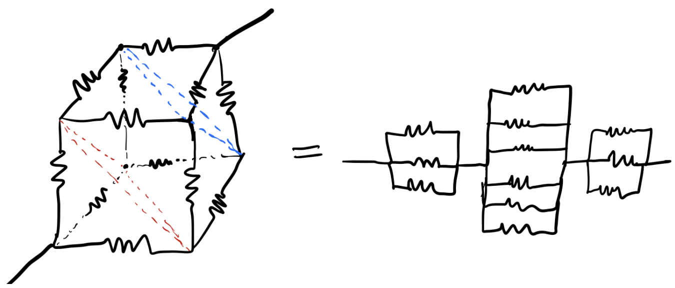
The resistances of the top/bottom branches are then both (adding in series):

$$R_{t/b} = R_1 + R_2 \tag{1.1}$$

So the total resistance of the configuration is:

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_t} + \frac{1}{R_b} = \frac{2}{R_1 + R_2} \implies \boxed{R_{\text{eq}} = \frac{R_1 + R_2}{2}} \tag{1.2}$$

Back to the cube question. In the last question, we removed a connection via an equipotential argument. The reverse is also true; we can freely add wires between equipotential points. Identifying intermediate vertices as equipotentials, we can connect the cube up as follows, giving the equivalent circuit:



The resistance of the first/third portions are:

$$\frac{1}{R_{1/3}} = \frac{1}{R} + \frac{1}{R} + \frac{1}{R} \implies R_{1/3} = \frac{R}{3} \tag{1.3}$$

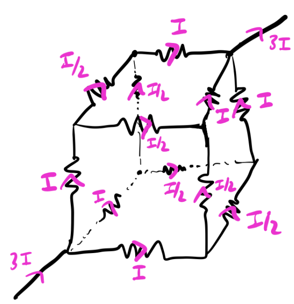
and the resistance of the middle portion is:

$$\frac{1}{R_2} = \frac{1}{R} + \frac{1}{R} + \frac{1}{R} + \frac{1}{R} + \frac{1}{R} + \frac{1}{R} \implies R_2 = \frac{R}{6} \tag{1.4}$$

so the total resistance of the cube is:

$$\boxed{R_{\text{cube}} = R_1 + R_2 + R_3 = \frac{5R}{6}} \tag{1.5}$$

Another way to solve this problem; consider injecting current $3I$ through the cube. By symmetry, current splits evenly at each intermediate node, as below:



Any path we take through the cube, the voltage drop is:

$$V = IR + \frac{I}{2}R + IR = \frac{5}{2}IR \tag{1.6}$$

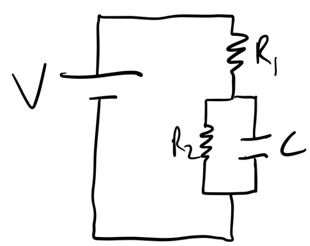
So to find the equivalent resistance of the cube, we divide this by the input current of $3I$:

$$\boxed{R_{\text{cube}} = \frac{\frac{5}{2}IR}{3I} = \frac{5}{6}R} \tag{1.7}$$

which agrees with our earlier analysis.

2 Series-Parallel RC Circuit

Consider the following circuit:



At time $t = 0$, there is no charge on the capacitor, and we close the switch on the circuit. The capacitor then is charged in time via the work done by the battery.

(a) What is the maximum current I_{max} that charges the capacitor (over all times)?

(b) What is the charge stored in the capacitor at $t = \infty$?

The solution will require us to think about how a capacitor behaves in a circuit when it is fully charged vs. not charged at all.

(a) The current through the capacitor is maximized at $t = 0$, before there is any charge on it. At $t = 0$, since the capacitor is uncharged, it acts as a wire without resistance. Thus, all current goes through the path with the capacitor, and R_2 can be ignored. At this time, the resistance of the circuit is just R_1, and so the current through the capacitor is:

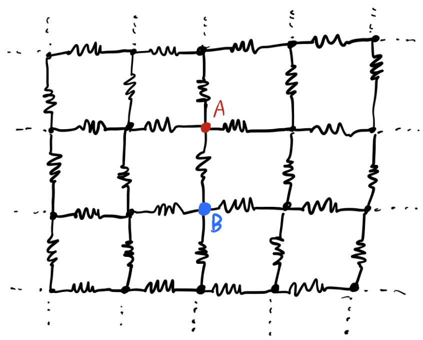
$$\boxed{I_{\text{max}} = \frac{V}{R_1}} \tag{2.1}$$

(b) At $t = \infty$, the battery is no longer capable of doing any more work to charge the capacitor (it is maximally charged). Thus no more current flows through that branch of the circuit, i.e. the capacitor acts as a full break/infinite resistor. The total resistance of the circuit is thus just that of R_1, R_2 in series, so $R = R_1 + R_2$, with current $I = \frac{V}{R_1+R_2}$ through the circuit. The voltage drop across R_2 is then $V = IR_2 = \frac{VR_2}{R_1+R_2}$. The voltage drop across R_2 is the same as that across the capacitor, which then allows us to solve for the charge on the capacitor as:

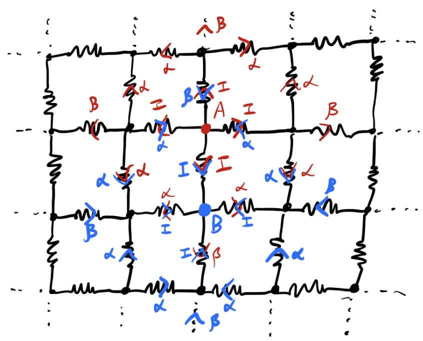
$$\boxed{Q = CV = \frac{CVR_2}{R_1 + R_2}} \tag{2.2}$$

3 Infinite Resistor Grid

Consider an infinite grid of resistors, each of resistance R. What is the equivalent resistance between adjacent nodes (say, between nodes A, B sketched below)?



We apply a symmetry argument to solve this. Consider injecting $4I$ current at node A, and withdrawing $4I$ current at node B, and superimposing the two configurations. By symmetry, the current through the resistor network looks like:

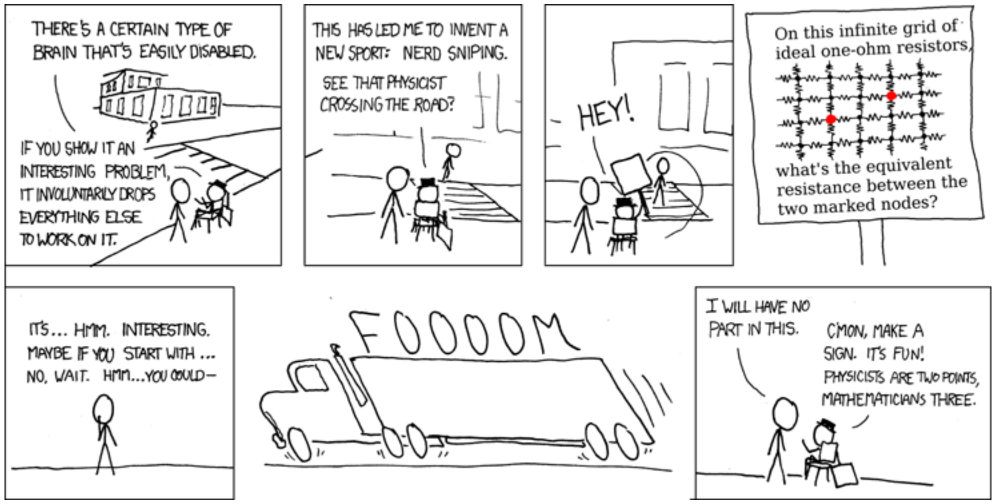


Looks complicated, but the main takeaway from this superposition is that $2I$ current flows through the resistor directly connecting A, B while if we sum up the current flowing from the other three nodes of A, we find $I + I + I - (2\alpha + \beta) = 2I$. Thus, the direct link carries $2I$ while the rest of the network carries $2I$, the same amount. Thus, the resistance of the entire rest of the network is equivalent to the resistor that joins A, B, which is just R. The equivalent resistance between the two nodes is then simply obtained by adding two resistors of R in parallel:

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R} + \frac{1}{R} \implies \boxed{R_{\text{eq}} = \frac{R}{2}} \tag{3.1}$$

Bonus question: What is the resistance between nodes that are a knight's move away from one another? As seen in the famous xkcd[1]:

[1] https://xkcd.com/356/



Fun little blogpost about this, for the curious. https://www.mathpages.com/home/kmath668/kmath668.htm. Certainly outside of the scope of this course.