PHYS 142 Discussion Week 4 - Midterm Review

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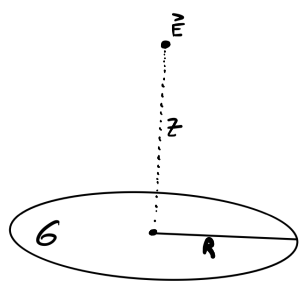
This document was typeset on February 5, 2025

1 Charged Circular Plate - Electric Field (and when/when not to use Gauss' Law)

Central question we shall discuss: what is the force felt by a charge q a distance z above a circular plate (through the central axis) of uniform surface charge density σ and radius R?



To solve this, we have to find the electric field at the given point. At this point, we may ask if we can apply Gauss' law:

$$\frac{\int \rho dV}{\epsilon_0} = \frac{Q_{\text{encl}}}{\epsilon_0} = \int_{\partial V} \mathbf{E} \cdot d\mathbf{A} \tag{1.1}$$

or not - we certainly can (as a mathematical statement it is always true!) but it will *not* be useful. For solving electrostatics problems, Gauss' law is useful for solving for electric fields when high symmetry allows us to easily evaluate the flux integral on the RHS as $|\mathbf{E}|$ times the surface area of the surface. This is true for spherical, infinite uniformly chargeed cylinders, and infinite uniformly charged plates (examples you have done, probably multiple times). But here we have a finite disk - and so we do not have such a strong symmetry here that allows us to use Gauss' law to easily solve for the $\mathbf{E}$ field - we have to explicitly calculate the integral over the charge distribution instead (however, we do recover strong symmetries in certain limits. What are these?).

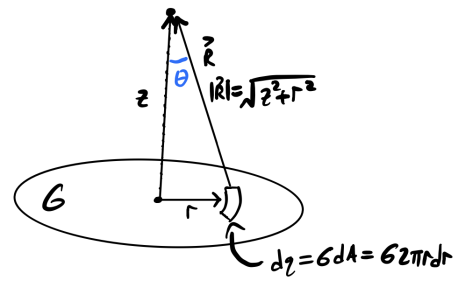
Going through the calculation (with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$):

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{R}|^2} \hat{\mathbf{R}} d\tau = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{|\mathbf{R}|^2} \hat{\mathbf{R}} dA \tag{1.2}$$

Symmetry is still useful for this integral. Via radial symmetry, the field must point in the $\hat{\mathbf{z}}$ direction (all radial parts are cancelled by charges on opposing sides of the disc). So we need only calculate $E_z = \mathbf{E} \cdot \hat{\mathbf{z}}$. A given amount of charge at radius r is given by:

$$dq = \sigma(\mathbf{r}) dA = \sigma 2\pi r dr \tag{1.3}$$

and $|\mathbf{R}|$ for dq at r at the point $z\hat{\mathbf{z}}$ is given by $\mathbf{R} = \sqrt{R^2 + z^2}\hat{\mathbf{R}}$ with the z-component given by given by $\hat{\mathbf{R}} \cdot \hat{\mathbf{z}} = \cos\theta = \frac{z}{\sqrt{z^2+r^2}}$.



Thus our integral reduces to:

$$\mathbf{E} = E_z \hat{\mathbf{z}} = \left(\frac{1}{4\pi\epsilon_0}\int_0^R \frac{\sigma 2\pi r dr}{r^2+z^2}\hat{\mathbf{R}}\cdot\hat{\mathbf{z}}\right)\hat{\mathbf{z}} = \frac{\hat{\mathbf{z}}}{4\pi\epsilon_0}\int_0^R \frac{\sigma 2\pi r dr}{4\pi\epsilon_0}\frac{z}{\sqrt{r^2+z^2}} = \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\int_0^R \frac{rdr}{(r^2+z^2)^{3/2}} = \frac{\sigma z}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right)\hat{\mathbf{z}} \tag{1.4}$$

where the last integral is solved via u-substitutiton of $u = r^2 + z^2$. The force is then obtained via $\mathbf{F} = q\mathbf{E}$.

2 Charged Circular Plate - Limits (and r dependence of common charge distributions)

A good sanity check with solving physics problems (and indeed sometimes this is part of exam problems) is to make sure that your result makes sense in specific limits. Here we can check that our result makes sense in the limits of $z \ll R$ and $z \gg R$.

- $z \ll R$: In this limit we are very close to the plate. Thus we can neglect the $\frac{1}{\sqrt{z^2+R^2}}$ term, wherein the field reduces to:

  $$\mathbf{E} \approx \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}} \tag{2.1}$$

  which is the electric field of an infinite uniform charged plate; this makes sense, as close to the plate it looks effectively infinite, and we reproduce the infinite charged plate result that we calculated previously using Gauss' Law.

- $z \gg R$: In this limit we are very far away from the plate. We taylor expand (binomial expansion is very common and a good one to know, where $(1+\epsilon)^n \approx 1 + n\epsilon + \frac{n(n-1)}{2!}\epsilon^2$ for small ϵ):

  $$\frac{1}{\sqrt{z^2+R^2}} = \frac{1}{z}\frac{1}{\sqrt{1+\frac{R^2}{z^2}}} \approx \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right) \tag{2.2}$$

  So then:

  $$\mathbf{E} \approx \frac{\sigma z}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right)\right)\hat{\mathbf{z}} = \frac{\sigma R^2}{4\epsilon_0}\frac{1}{z^2}\hat{\mathbf{z}} = \frac{\sigma\pi R^2}{4\pi\epsilon_0}\frac{1}{z^2}\hat{\mathbf{z}} \tag{2.3}$$

  Which is the electric field from a point charge of charge $Q = \sigma\pi R^2$, as we would expect.

Different charge distributions have different dependencies of the electric field; we list some common ones (and how you would find this dependence) below.

Charge distribution	r-dependence	How to find this
Inside of a uniform volume charge distribution	r^1	Gauss' Law
Outside of an infinite plane	r^0	Gauss' Law
Infinite line/Outside an infinite cylindrical distribution	r^{-1}	Gauss' Law
Point Charge/Outside a spherical distribution	r^{-2}	Gauss' Law
Dipole	r^{-3}	Superposition + Taylor expansion
Quadrapole	r^{-4}	Superposition + Taylor expansion

3 Charged Circular Plate - Potential (and why is potential well-defined)

What is the potential $\phi(z)$ at a distance z above the plate (on the symmetry axis)? We can obtain this via the path integral of the electric field (taking $\phi = 0$ at infinity). In particular we choose a path that starts at $L\hat{\mathbf{z}}$ and goes straight down to $z\hat{\mathbf{z}}$:

$$\begin{aligned}
\phi(\mathbf{r}) &= -\int_L^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{r} \\
&= -\int_L^z \frac{\sigma z}{2\epsilon_0} \left(\frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}} \right) dz \\
&= -\frac{\sigma}{2\epsilon_0} \int_L^z \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \\
&= \frac{\sigma}{2\epsilon_0} \left(\sqrt{z^2 + R^2} - z - \sqrt{R^2 + L^2} + L \right) \\
&\overset{L \to \infty}{\to} \frac{\sigma}{2\epsilon_0} \left(\sqrt{z^2 + R^2} - z \right)
\end{aligned} \tag{3.1}$$

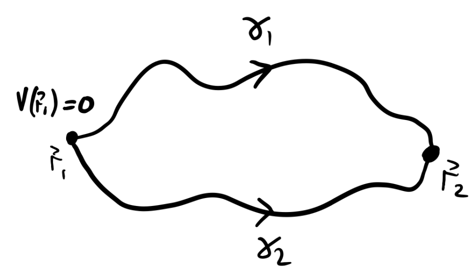
You could also get this via integrating the charge distribution (and the electric field could be obtained as the gradient $\mathbf{E} = -\nabla\phi$) - this would be a good exercise to check the consistency of your result! And actually may be an easier way to do this calculation, since the integral is of a scalar function.

We chose a particular path in the above calculation, but indeed we could have chose any path and gotten the same answer. Indeed, this is a defining quality of the potential function (path independence of the potential differences). What property of the electric field makes this so?

Indeed, and this is indeed a defining quality of the potential function (path independence of potential differences). What property of the electric field makes this so? Namely it comes from the fact that the electric field is conservative/curl free:

$$\nabla \times \mathbf{E} = 0. \tag{3.2}$$

Let's prove why this is so. Suppose we compute the potential $\phi(\mathbf{r}_2)$ via two different paths:



Wherein:

$$\phi_{\gamma_1}(\mathbf{r}_2) = -\int_{\gamma_1} \mathbf{E} \cdot d\mathbf{r}, \quad \phi_{\gamma_2}(\mathbf{r}_2) = -\int_{\gamma_2} \mathbf{E} \cdot d\mathbf{r} \tag{3.3}$$

The difference between the potential as evaluated on the two paths is the integral over the closed path $\gamma_2 - \gamma_1$:

$$\phi_{\gamma_2}(\mathbf{r}_2) - \phi_{\gamma_1}(\mathbf{r}_2) = -\oint_{\gamma_2-\gamma_1} \mathbf{E} \cdot d\mathbf{r} \tag{3.4}$$

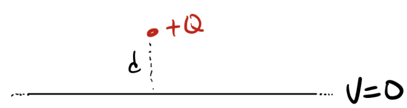
By Stokes' theorem, this is equal to the surface integral of $\nabla \times \mathbf{E}$ on the surface S enclosed by $\gamma_2 - \gamma_1$:

$$\phi_{\gamma_2}(\mathbf{r}_2) - \phi_{\gamma_1}(\mathbf{r}_2) = -\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{A} = 0 \tag{3.5}$$

where in the last equality we conclude that the difference vanishes due to the curl of the electric field vanishing. Thus, the potential is indeed path-independent, coming from the fact that the electric field is curl free!

4 Conducting Circular Plate - Poisson's Equation and Method of Images

Suppose the plate is now a grounded, neutral conductor. Is there a limit in which we can still solve for the force acting on the charge? Indeed, we can solve for this in the $d \ll L$ limit where the plane looks infinite, via the method of images.



Namely, we leverage the fact that the Poisson equation:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{4.1}$$

satisfies the uniqueness theorem - namely, if for a given charge distribution ρ on a domain D, if we fix the potentials on the boundaries of D then the solution is unique.

A quick primer for how we derived the Poisson equation; we start with Gauss' Law:

$$\frac{\int \rho dV}{\epsilon_0} = \int_{\partial V} \mathbf{E} \cdot d\mathbf{A} \tag{4.2}$$

Applying the divergence theorem on the RHS:

$$\frac{\int \rho dV}{\epsilon_0} = \int \nabla \cdot \mathbf{E} dV \tag{4.3}$$

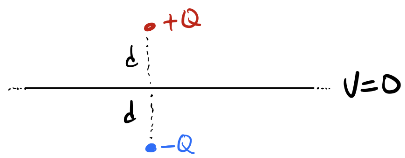
this holds for an arbitrary volume, so we obtain the differential form of Gauss' law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4.4}$$

Then substituting in $\mathbf{E} = -\nabla \phi$ we obtain:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}. \tag{4.5}$$

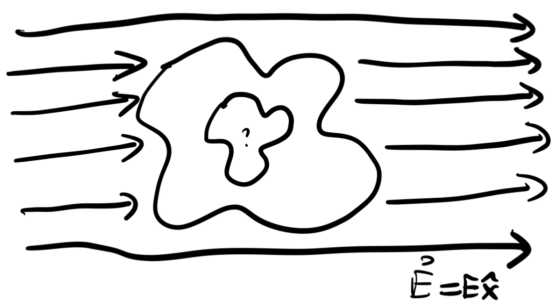
Back to the question at hand - we want to leverage the uniqueness theorem, therein we want to find a charge distribution that matches the $\phi = 0$ boundary condition we have on the xy-plane/on the conductor. Therein, we realize that a simple charge configuration that achieves $\phi = 0$ is simply placing an "image charge" of equal and opposite charge $-q$ on the opposite side of the conducting plate.



We thus obtain that the potential, and thus electric field and force on the positive charge (which physically comes from the induced negative charges on the conducting plate) is the same as that as arising from a negative image charge placed on the opposite side of the conducting plate. (See discussion 3 for more details on the method of images).

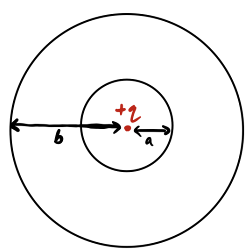
5 Conductors Properties

Consider a conductor with a cavity (with no charge) inside, in the presence of an external electric field $\mathbf{E} = E\hat{\mathbf{x}}$. Is there any electric field inside the conductor? Inside the cavity?



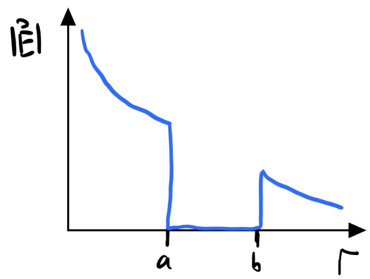
The answer is no for both! Inside the conductor, there are no $\mathbf{E}$-fields (as the charges rearrange themselves in such a way that there are no net electric fields inside the conductor at equilibrium). Further, there are no electric fields inside the cavity. Why? Since there is no $\mathbf{E}$ inside the conductor, it follows the entire conductor is at some constant potential $\phi = \phi_0$. Thus, inside the cavity, we have no charge, and we have $\phi = \phi_0$ on the boundary everywhere. The only solution to Laplace's equation is then the constant $\phi = \phi_0$ everywhere inside the cavity and hence $\mathbf{E} = -\boldsymbol{\nabla}\phi = 0$.

We can also consider the following system, where we have a charge $+q$ at the origin, and a neutral conducting spherical shell from $r = a$ to $r = b$. What does the $\mathbf{E}$-field strength look like as a function of r?



Inside of the cavity, it is clear that we just have the $\frac{1}{r^2}$ electric field from the point charge. Inside of the conductor, we have zero ($\mathbf{E} = \mathbf{0}$ in a conductor - in particular, $-q$ negative charge spreads itself out on the inner surface to cancel out the $\mathbf{E}$-field from the positive charge at the center). Outside of the conductor, we have that $+q$ charges spreads itself out symmetrically on the outer surface (the conductor is electrically neutral, and we have $+q$ charge that neutralizes the $-q$ charge on the inner surface - this

$+q$ charge spreads itself on the outer surface of the conductor, which is the only place/distribution it can have to get $\mathbf{E} = 0$ inside the conductor). By Gauss' law, we just have the $\frac{1}{r^2}$ electric field from a $+q$ point charge once again.



Also, the property of conductors imply that even if we were to move the $+q$ charge inside of the hollow cavity, the field outside of the conductor would not change. Can you explain why?

6 Capacitance and Energy

Let's derive the capacitance $C = \frac{Q}{\phi}$ for a couple different charge configurations. Physically, it is a measure of how much charge per potential difference there is - this is highly dependent on geometry. Let's see how it differs for parallel plates, spheres, cylinders considering $\pm Q$ on the two parts.

Parallel plates with area A (put one $(+Q)$ plate on the xy-plane with $z = 0$, put other plate at $z = d$) have surface charge density $\sigma = \frac{Q}{A}$, and electric field from one plate (From Gauss' Law) is $\frac{Q}{2A\epsilon_0}\hat{\mathbf{z}}$, superimposing both we have $\frac{Q}{A\epsilon_0}\hat{\mathbf{z}}$. So, the potential difference between the plates is:

$$\Delta\phi = \phi(\mathbf{0}) - \phi(d\hat{\mathbf{z}}) = -\int_{d\hat{\mathbf{z}}\to\mathbf{0}} \mathbf{E}\cdot d\mathbf{r} = -\int_d^0 \frac{Q}{A\epsilon_0}dz = \frac{Q}{A\epsilon_0}d \tag{6.1}$$

Thus the capacitance is:

$$C = \frac{Q}{\phi} = \frac{A\epsilon_0}{d} \tag{6.2}$$

Two concentric spherical shells with radii a, b. In this case Gauss' law tells us that the field in between the two spheres is radially outwards, with:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{6.3}$$

for $a < r < b$. Thus the potential difference is:

$$\Delta\phi = \phi(a\hat{\mathbf{r}}) - \phi(b\hat{\mathbf{r}}) = -\int_{b\hat{\mathbf{r}}\to a\hat{\mathbf{r}}} \mathbf{E}\cdot d\mathbf{r} = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2}dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \tag{6.4}$$

Thus the capacitance is:

$$C = \frac{Q}{\phi} = 4\pi\epsilon_0\frac{ab}{b-a} \tag{6.5}$$

Two concentric cylinders with radii a, b. Suppose the charge per unit length L is Q. In this case Gauss' law tells us that the field in between the two cylinders (the only place where it is nonzero) is radially outwards, with:

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 Lr}\hat{\mathbf{r}} \tag{6.6}$$

for $a < r < b$. The potential difference is:

$$\Delta\phi = \phi(a\hat{\mathbf{r}}) - \phi(b\hat{\mathbf{r}}) = -\int_{b\hat{\mathbf{r}}\to a\hat{\mathbf{r}}} \mathbf{E}\cdot d\mathbf{r} = -\int_b^a \frac{Q}{2\pi\epsilon_0 Lr}dr = \frac{Q}{2\pi\epsilon_0 L}\ln(\frac{b}{a}) \tag{6.7}$$

Thus the capacitance (per unit length L) is:

$$C = \frac{1}{L}\frac{Q}{\phi} = 2\pi\epsilon_0 \ln(\frac{b}{a}) \tag{6.8}$$

What is the energy stored in a capacitor? Consider moving dq charge against potential difference $\phi = \frac{q}{C}$:

$$dW = \phi dq = \frac{q}{C}dq \tag{6.9}$$

Then if we integrate from initial (0) to final charge:

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\phi^2 \tag{6.10}$$

Let's see that the energy U stored in a parallel plate capacitor checks out, using three methods.

- The above formula based on capacitance:

$$W = \frac{1}{2}C\phi^2 = \frac{1}{2}\frac{A\epsilon_0}{d}\cdot\left(\frac{Q}{A\epsilon_0}d\right)^2 = \frac{Q^2 d}{2A\epsilon_0} \tag{6.11}$$

- Calculated by integrating over the electric field:

$$U = \frac{\epsilon_0}{2}\int \mathbf{E}^2 d\phi = \frac{\epsilon_0}{2}\left(\frac{Q}{A\epsilon_0}\right)^2 A\cdot d = \frac{Q^2 d}{2A\epsilon_0} \tag{6.12}$$

- Calculating the work done to pull the plates apart:

$$W = \int \mathbf{F}\cdot d\mathbf{x} = Q\int \cdot\mathbf{E}\cdot d\mathbf{x} = Q\frac{Q}{2A\epsilon_0}d = \frac{Q^2 d}{2A\epsilon_0} \tag{6.13}$$