PHYS 142 Discussion Week 2 - Energy in Fields and Potentials

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1 Problem Statement

This discussion is based on Purcell 1.24 and 1.83.

In your quiz this week, you studied an infinitely long solid cylinder of radius a and uniform positive charge density per unit volume of $+\rho$. You found the electric field (choosing a coordinate system such that the center of the cylinder is aligned with the z-axis) using Gauss' Law to be:

$$\mathbf{E}(r) = \begin{cases} \frac{\rho r}{2\epsilon_0}\hat{\mathbf{r}} & r \leq a \\ \frac{\rho a^2}{2\epsilon_0 r}\hat{\mathbf{r}} & r > a. \end{cases} \tag{1.1}$$

In this discussion, we will find the energy stored in this charge configuration/the electric fields (per unit length) using two methods:

(i) Consider building up the cylinder layer by layer by bringing charges from a faraway radius R (if we bring the charges in from infinity, we will find that the energy diverges), and adding up the energy contributions from each layer.

(ii) By evaluating the integral $U = \frac{\epsilon_0}{2}\int \mathbf{E}^2 dV$ over all space (again, consider integrating out to some large radius R).

2 Solution

(i) For the first method, we consider that at an intermediate stage, the cylinder has radius r; the electric field at this point at radius r' is:

$$\mathbf{E}(r') = \frac{\rho r^2}{2\epsilon_0 r'}\hat{\mathbf{r}} \tag{2.1}$$

Therefore, the work done in bringing a charge dq from radius R to radius r is:

$$dW = -\int_R^r dq\mathbf{E}\cdot d\mathbf{r} = -\int_R^r dq\frac{\rho r^2}{2\epsilon_0 r'}dr' = dq\frac{\rho r^2}{2\epsilon_0}\ln(\frac{R}{r}) \tag{2.2}$$

Building up the cylinder, we are bringing in cylindrical shells of charge:

$$dq = (2\pi r dr)l\rho \tag{2.3}$$

Thus the total work done in bringing in shells of radii $r = 0$ to $r = a$ is:

$$W = \int dW = \int_0^a dq\frac{\rho r^2}{2\epsilon_0}\ln(\frac{R}{r}) = \int_0^a \frac{(2\pi r dr)l\rho\rho r^2}{2\epsilon_0}\ln(\frac{R}{r}) = \frac{\pi\rho^2 l}{\epsilon_0}\int_0^a r^3\ln(\frac{R}{r})dr \tag{2.4}$$

The integral can be solved via integration by parts, with $u = \ln(\frac{R}{r})$ with $du = -\frac{1}{r}$ and $dv = r^3$, $v = \frac{r^4}{4}$:

$$\int r^3\ln(\frac{R}{r})dr = \frac{r^4}{4}\ln(\frac{R}{r}) - \int\frac{r^4}{4}\left(-\frac{1}{r}\right)dr = \frac{r^4}{4}\ln(\frac{R}{r}) + \frac{r^4}{16} \tag{2.5}$$

So then:

$$W = \frac{\pi\rho^2 l}{\epsilon_0}\left(\frac{r^4}{4}\ln(\frac{R}{r}) + \frac{r^4}{16}\bigg|_0^a\right) = \frac{\pi\rho^2 a^4 l}{4\epsilon_0}\left(\ln(\frac{R}{a}) + \frac{1}{4}\right) \tag{2.6}$$

The total work done is equal to the potential energy stored in the configuration, so dividing this out by l (to get the energy per unit length), we conclude:

$$\boxed{U = \frac{\pi\rho^2 a^4}{4\epsilon_0}\left(\ln(\frac{R}{a}) + \frac{1}{4}\right)} \tag{2.7}$$

(ii) For the second method, we evaluate the volume integral:

$$U = \frac{\epsilon_0}{2}\int_{\text{field}} \mathbf{E}^2 dV. \tag{2.8}$$

We have already found the field, so all is left to carry this out. Consider a length l of the cylinder. The energy stored in the field from radius $r = a$ to $r = R$ (outside of the cylinder) is thus:

$$U_{\text{ext}} = \frac{\epsilon_0}{2}\int_0^{2\pi} d\varphi \int_0^l dz \int_a^R \left(\frac{\rho a^2}{2\epsilon_0 r}\hat{\mathbf{r}}\right)^2 r dr = \frac{\epsilon_0}{2}2\pi l \int_a^R \frac{\rho^2 a^4}{4\epsilon_0^2 r^2} r dr = \frac{\pi\rho^2 a^4 l}{4\epsilon_0}\int_a^R \frac{dr}{r} = \frac{\pi\rho^2 a^4 l}{4\epsilon_0}\ln(\frac{R}{a}) \tag{2.9}$$

The energy stored in the field from radius $r = 0$ to $r = a$ (inside of the cylinder) is:

$$U_{\text{int}} = \frac{\epsilon_0}{2}\int_0^{2\pi} d\varphi \int_0^l dz \int_0^a \left(\frac{\rho a^2}{2\epsilon_0 r}\hat{\mathbf{r}}\right)^2 r dr = \frac{\pi\rho^2 l}{4\epsilon_0}\int_0^a r^3 dr = \frac{\pi\rho^2 a^4 l}{16\epsilon_0} \tag{2.10}$$

The total energy stored in the field per unit length is thus:

$$\boxed{U = \frac{U_{\text{ext}} + U_{\text{int}}}{l} = \frac{\pi\rho^2 a^4}{4\epsilon_0}\left(\ln(\frac{R}{a}) + \frac{1}{4}\right)} \tag{2.11}$$

3 Extra: Electric potential

The electric potential can also be derived easily from the electric field; taking $\phi(\mathbf{r} = \mathbf{0}) = 0$, we have (for $|\mathbf{r}| = r < a$):

$$\phi(r) = -\int_{\mathbf{0}\to\mathbf{r}} \mathbf{E}\cdot d\mathbf{r} = -\int_0^r E dr = -\int_0^r \frac{\rho r}{2\epsilon_0} dr = -\frac{\rho r^2}{4\epsilon_0} \tag{3.1}$$

and for $|\mathbf{r}| = r > a$:

$$\rho(r) = -\int_{\mathbf{0}\to a\hat{\mathbf{r}}} \mathbf{E}\cdot d\mathbf{r} - \int_{a\hat{\mathbf{r}}\to r\hat{\mathbf{r}}} = -\int_0^a \frac{\rho r}{2\epsilon_0} dr - \int_a^r \frac{\rho a^2}{2\epsilon_0 r} dr = -\frac{\rho a^2}{4\epsilon_0} - \frac{\rho a^2}{2\epsilon_0}\ln(\frac{r}{a}) \tag{3.2}$$