PHYS 142 Discussion Week 1 - Superposition, Symmetries, and Gauss' Law

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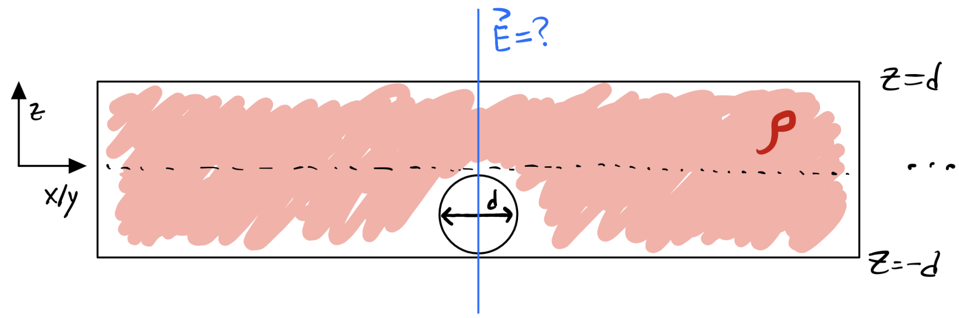
In this discussion, we'll be trying to solve the following, difficult seeming question:

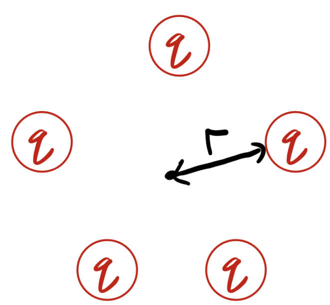
Suppose we have a infinite slab of electric charge of thickness $2d$ and charge density ρ. The slab extends in the x and y directions, being bisected by the xy-plane. From this slab we bore a spherical ball of radius $d/2$ centered at $(0, 0, -d/2)$. What is the electric field along the z axis?



This problem seems very hard to deal with! It seems like the electric field might do very complicated things as a result of removing the ball. But we will build up the tools necessary to solve this problem, by solving three easier problems in sequence.

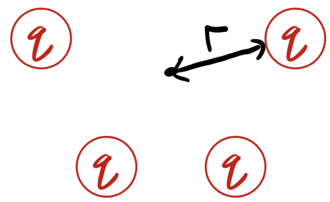
1 The charge pentagon

Suppose we place 5 charges of charge q at the edges of a pentagon, each at distance d from the center. What is the electric field in the center? Do we have to do a calculation to find out, or no?



The answer is zero by discrete (5-fold) rotational symmetry (and this does not require calculation!); if we propose any direction the field could point, there are 4 other directions where it could be equivalently be expected to point; but since none of these are preferred, so it must be the case that there is no field in the center at all!

Suppose we now remove the top charge; what is the electric field in the center? Do we have to do a calculation to find out, or no?

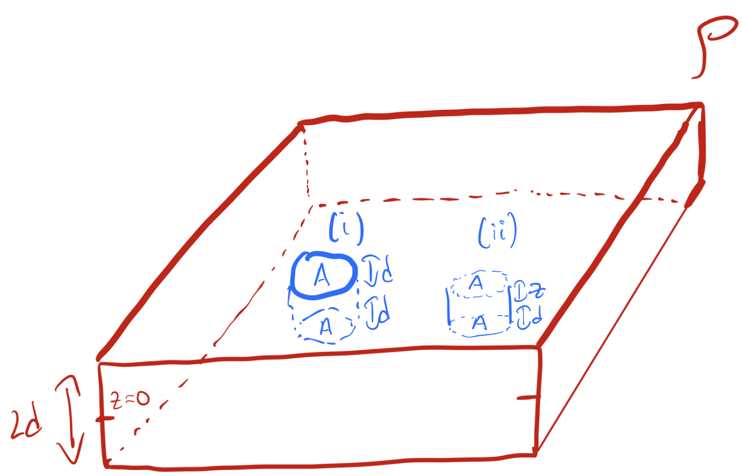


The answer is that the electric field in the center is $\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{z}}$ (again, this requires no calculation)! We can see this immediately from the principle of superposition. Removing the $+q$ charge is exactly the same as if we had *added* a further $-q$ charge to the top of the pentagon. So, we can take our previous zero-field result and simply superimpose the electric field from a single $-q$ charge placed at the top of the pentagon.

In fact, the arguments we have made here generalize to a charge n-gon for arbitrary $n \in \mathbb{Z}$.

2 Slab of uniform charge density

Let us solve for the electric field of just the slab. Since the system is translationally invariant in x and y, the electric fields can only possibly point in the $\hat{\mathbf{z}}$ direction. A natural way to solve for the electric field is thus to use a Gaussian pillbox with faces normal to the electric field.



First, we consider Gaussian surface (i), for which the relevant faces (for which the $\mathbf{E}$-field passes through) are above and below the slab. Both faces receive equal flux, and the surface contains charge $Q_{\text{encl}} = \rho V = \rho(A2d)$, so by Gauss' Law:

$$\frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\rho A2d}{\epsilon_0} = \int \mathbf{E} \cdot d\mathbf{A} = |\mathbf{E}|2A \implies |\mathbf{E}| = \frac{\rho d}{\epsilon_0} \tag{2.1}$$

And so above/below the slab we have:

$$\mathbf{E} = \begin{cases} \frac{\rho d}{\epsilon_0}\hat{\mathbf{z}} & z > d \\ -\frac{\rho d}{\epsilon_0}\hat{\mathbf{z}} & z < -d \end{cases} \tag{2.2}$$

What about within the slab? Well, now consider the Gaussian surface (ii), where one face is below the slab and the other face is in the slab at height z. Then, the surface contains charge $Q_{\text{encl}} = \rho V = \rho A(d+z)$, and we know that the bottom face has flux $\frac{\rho d}{\epsilon_0}A$, so:

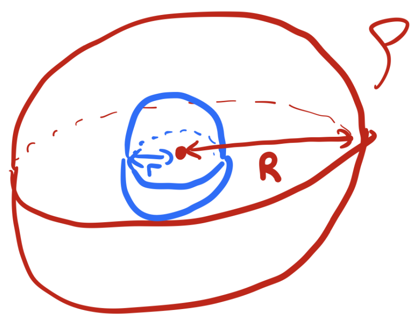
$$\frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\rho A(d+z)}{\epsilon_0} = \int \mathbf{E}\cdot d\mathbf{A} = \frac{\rho d}{\epsilon_0}A + |\mathbf{E}_{\text{in}}|A \implies |\mathbf{E}|_{\text{in}} = \frac{\rho z}{\epsilon_0} \tag{2.3}$$

Thus we also obtain the **E**-field inside the slab:

$$\mathbf{E} = \begin{cases} \frac{\rho d}{\epsilon_0}\hat{\mathbf{z}} & z > d \\ \frac{\rho z}{\epsilon_0}\hat{\mathbf{z}} & -d < z < d \\ -\frac{\rho d}{\epsilon_0}\hat{\mathbf{z}} & z < -d \end{cases} \tag{2.4}$$

3 Sphere of uniform charge density

Next, we consider the electric field arising from a sphere of uniform charge density ρ. Since the system is rotationally symmetric about the center of the sphere (which we choose to be the origin), the electric field from this configuration must point radially outwards. A natural way to solve for the magnitude is then to use a spherical Gaussian surface.



Consider first a spherical Gaussian surface of $r < R$. Then, the enclosed charge is $Q_{\text{encl}} = \rho V = \rho\frac{4}{3}\pi r^3$, so by Gauss' Law:

$$\frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\rho\frac{4}{3}\pi r^3}{\epsilon_0} = \int \mathbf{E}\cdot d\mathbf{A} = |\mathbf{E}|4\pi r^2 \implies |\mathbf{E}| = \frac{\rho r}{3\epsilon_0} \tag{3.1}$$

For $r > R$, we have that the entire sphere is enclosed, with $Q_{\text{encl}} = \rho V = \rho\frac{4}{3}\pi R^3$, so by Gauss' Law:

$$\frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\rho\frac{4}{3}\pi R^3}{\epsilon_0} = \int \mathbf{E}\cdot d\mathbf{A} = |\mathbf{E}|4\pi r^2 \implies |\mathbf{E}| = \frac{\rho R^3}{3\epsilon_0 r^2} \tag{3.2}$$

Thus:

$$\mathbf{E} = \begin{cases} \frac{\rho r}{3\epsilon_0}\hat{\mathbf{r}} & r < R \\ \frac{\rho R^3}{3\epsilon_0 r^2}\hat{\mathbf{r}} & r > R \end{cases} \tag{3.3}$$

4 Putting it together

Let's put it together to solve the original problem! We learned in the charge pentagon problem that removing a positive charge is equivalent to adding a negative charge to cancel it out. So, in our original problem, we can take the electric field from the full slab (as we found in problem 2) and add the electric field from a sphere of equal and opposite uniform charge density (as we found in problem 3) to find the total electric field! We don't have to do any new or complicated calculations. The slab electric field we found to be:

$$\mathbf{E}_{\text{sphere}}(z) = \begin{cases} \frac{\rho d}{\epsilon_0}\hat{\mathbf{z}} & z > d \\ \frac{\rho z}{\epsilon_0}\hat{\mathbf{z}} & -d < z < d \\ -\frac{\rho d}{\epsilon_0}\hat{\mathbf{z}} & z < -d \end{cases} \tag{4.1}$$

And for a sphere of radius $d/2$ and uniform charge density $-\rho$ centered at $(0, 0, -d/2)$, we have (along the z axis):

$$\mathbf{E}_{\text{sphere}}(z) = \begin{cases} -\frac{\rho(z+\frac{d}{2})}{3\epsilon_0}\hat{\mathbf{z}} & -d < z < 0 \\ -\frac{\rho d^3}{24\epsilon_0\left(z+\frac{d}{2}\right)^2}\hat{\mathbf{z}} & z > 0 \;\&\; z < -d \end{cases} \tag{4.2}$$

Thus the total electric field on the z-axis we obtain via superposition:

$$\mathbf{E}_{\text{tot}} = \begin{cases} \left(\frac{\rho d}{\epsilon_0} - \frac{\rho d^3}{24\epsilon_0\left(z+\frac{d}{2}\right)^2}\right)\hat{\mathbf{z}} & z > d \\ \left(\frac{\rho z}{\epsilon_0} - \frac{\rho d^3}{24\epsilon_0\left(z+\frac{d}{2}\right)^2}\right)\hat{\mathbf{z}} & 0 < z < d \\ \left(\frac{\rho z}{\epsilon_0} - \frac{\rho(z+\frac{d}{2})}{3\epsilon_0}\right)\hat{\mathbf{z}} & -d < z < 0 \\ \left(-\frac{\rho d}{\epsilon_0} - \frac{\rho d^3}{24\epsilon_0\left(z+\frac{d}{2}\right)^2}\right)\hat{\mathbf{z}} & z < -d \end{cases} \tag{4.3}$$