PHYS 366 (Advanced Condensed Matter Physics) Notes

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Introduction:
This is a set of lecture notes taken from UChicago's PHYS 366 (Advanced Condensed Matter Physics), taught by Kathryn Levin. Topics covered include Magnetism, Mean Field Theory, Second Quantization, The Coloumb Gas, The Hubbard Model, BCS Theory, Bose Gases, Bose Superfluids, Landau-Gnsberg Theory, Gross-Pitaevskii Theory, The Proximity Effect, The Meissner Effect, The Josephson Effect.

Contents

1 Course Overview + Magnetism

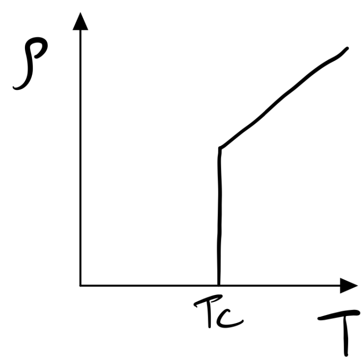
1.1 Course Overview

Topics

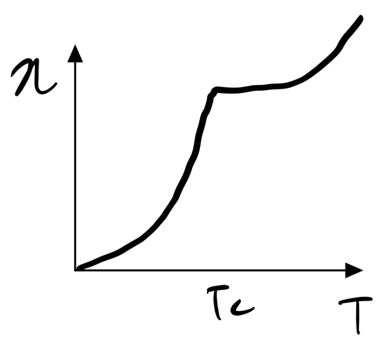
In this course we will discuss superconductivity, superfluidity, magnetism - all under the umbrella of co-operative phenomena in solids/gases/fluids. The emphasis will not be on QFT, but rather on phenomenology.

Magnetism comes in two flavours; in itinerate (metallic) and localized (insulator) forms. Superfluidity also has two flavours, in that it can occur in bosons and in fermions. For a qualitative picture; if we have a particles in a trap, at $T = 0$ bosons all occupy the E_0 ground state (Bose-Einsetein condensation). For fermions, normally (via Pauli exclusion) occupy various energy levels. But in a fermionic superfluid, we have pairs of fermions which act as bosons and condense. But this requires an attraction/pairing mechanism of some kind.

A superconductor (which can only occur in fermions), we see a zero resistivity above a critical temperature T_c:



In superfluids, we have zero viscosity at $T = 0$, and a kink at T_c:



In this course we will discuss various families of matter:

- Conventional and exotic superconductors
- Families of Helium (He^3, He^4)
- Alkali gases, ultra cold atoms
- Non-tabletop superconductors (we won't discuss these in detail, but as interesting examples for the overview)

- Color superconductivity (quark-gluon plasma, with a $T_c \sim 10^{12}$K)
- Nuclear matter
- Neutron stars

- Hydrides under pressure (e.g. H_2S), false(!) claim of $H +$ Lutecium $+ N$, Moire materials, etc. many controversial materials. We can do a journal club type discussion of these as there are interest.

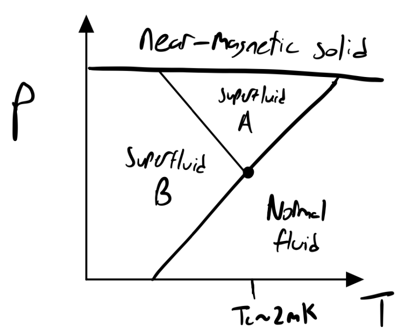
Exotic Superconductors

Types of exotic superconductors:

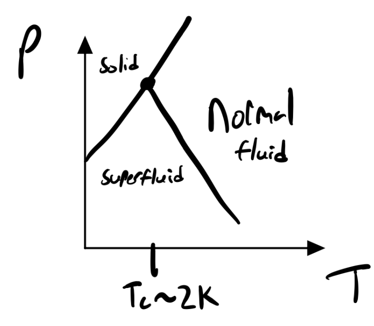
- Cuprates (CUO_2, $YBaCuO$) with $T_c \sim 100$K
- Fe-based with $T_c \sim 50$K
- Heavy fermions with $m^* \sim 1000 m_e$ with $T_c \sim 2$K
- Fullerenes (e,g. A_3C_{60}) with $T_c \sim 30$K
- Moire matrerials (e.g. WSe_2) dichalocquides of $T_c \sim 1 - 2$K

Helium

Helium-3 is a fermion, which obeys BCS-theory (BCS-triplet). Phenomenologically interesting as a new kind of pairing mechanism observed. Has the phase diagram:



For He-4 (boson), we instead have the phase diagram:

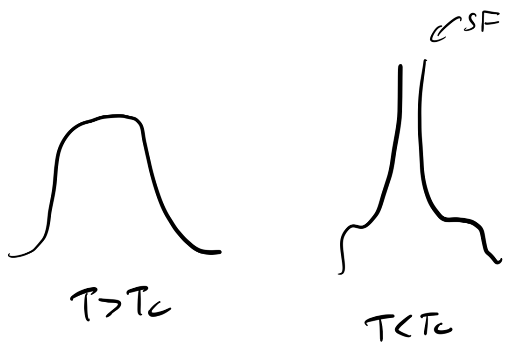


and there is no microscopic understanding of it to this day! Such a strongly interacting and difficult to analyze system.

Alkali Gases

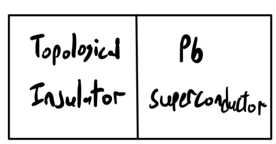
We have lithium in fermionic and bosonic forms Li^6 (1995) and Li^7 (2003). T_c is tiny; $T_c \sim 2\mu$K. To cool these, we use a 2-step process. First, we have a laser cooling process (where we fire the atoms with photons, which recoil and take away the energy). We then do evaporative cooling - which is “easy” for Bose gases, and hard for fermions. Why is it hard? Most efficient cooling mechanism is collisions via the s-wave channel, but fermions cannot do this nanturally; need to use an extra hyperfine state. There was a hope that this might give insight into He-4, but since Alkali gases are quite dilute/weakly interacting, there isn’t that much to gleam about strongly interacting He-4.

Experiments in this area are quite impressive - you can’t measure temperature directly. Instead, you measure density profiles, which tells you if a condensate/superfluid has formed.



Topological Superconductivity

This is the “holy grail” of condensed matter physics. There are two approaches; there is one approach which looks for an intrinsic superconductor, and there is another approach which uses the proximity effect. In the latter approach, we take something that is topological (e.g. an insulator) and put it next to a superconductor. Why might this be beneficial? We get $p + ip$ SC which hosts Majorana particles. For a long time Microsoft put a lot of money into this because they thought this could be a stable platform for quantum computation.



The long list of Nobels

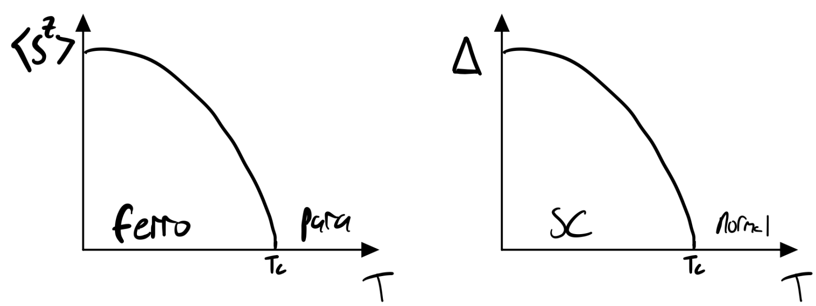
- 1913 - Onnes discovers superconductivity in mercury (well, his students measured it while he was on vacation...)
- 1972 - Bardeen Cooper Schrieffer for BCS theory that explains superconductors (fun historical side-note; apparently Feynman)
- 1973 - Josephson effect; current without voltage can happen (and this is used for, e.g., superconducting qubits) because cooper pairs can tunnel.
- 1975 - Bohr & Moltelson for pairing in nuclear physics

- 1978 - Kapitsa for discovery of He^4
- 1987 - Bednortz-Mueller for Cuprates
- 1996 - Oscheroff et al for He^3 superfluid
- 2001 - Kettere, Cornell, Weinman for alkali gas superfluids
- 2003 - Abrikosov and Leggett for theory (of He^3, vortices)
- 2008 - Nambu for taking ideas in CM - such as gauge symmetry breaking and applying them to high-energy physics.
- 2012 - Higgs. How does this relate to superconductivity? Well, SC wavefunction is $\psi = |\psi|e^{i\phi}$ for $\sim 10^{23}$ particles; all fixed phase. We call this a broken gauge symmetry. Without going into detail, the fluctuations in the amplitude $|\psi|$ connects to the Higgs boson.

Notably, Schrieffer, Jospheson, Oscheroff were all grad students when they made their discoveries!

1.2 Magnetism and symmetry-breaking

We mentioned above the broken gauge theory in the superconducting wavefunction. Let's start by looking at a more intuitive example first, looking at broken symmetry in localized magnets. For example, we can look at we look at $\langle S^z \rangle$ in a (conventional) magnet, and find spontaneously broken symmetry; system Hamiltonian has no preferred direction for the magnetization to point but the system picks one direction. How does this connect to superconductors? If we look at Δ (the binding energy) versus the temperature in a material that can host superconductivity, we have:



For a superconductor we need fermion pairs, and below a critical temperature we have that there is a finite binding energy leading to BCS pairs of condensed of fermions.

In this course we won't get into the nitty gritty of QFT and Green's functions. We will instead look at toy models of phonon induced attraction of the fermions.

1.3 Mean field theory as a solution to the many-body problem

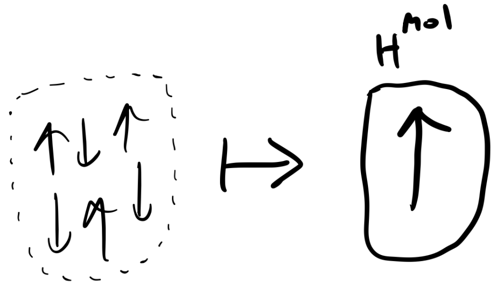
We might start with the quantum-mechanical Hamiltonian:

$$H = -\frac{1}{2}\sum_{ij} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j \tag{1.1}$$

where $\mathbf{S}_i$ is the localized spin; there is no kinetic energy and the spins don't go anywhere. We can also write down the itinerant Hamiltonian (in the second quantization language):

$$H = \sum_k \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + u\sum_{ij} c^\dagger_{i\sigma} c_{i\sigma} c^\dagger_{j\sigma} c_{j\sigma} \tag{1.2}$$

We can replace the interaction J_{ij} in the spin Hamiltonian with a moleculate field $H_{MF} = H^{mol} \sum_i S_i^z$. H^{mol} is found by self-consistent calculation. Pictorially, we imagine taking a bunch of interacting spins and replace it with a single spin under the influence of H^{mol}.



In terms of a recipe, we want to map:

$$\sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \to \sum_i H^{mol} S_i^z \tag{1.3}$$

where in the first step we replace:

$$\mathbf{S}_i \cdot \mathbf{S}_j \to \langle \mathbf{S}_i \rangle \mathbf{S}_j + \mathbf{S}_i \left\langle \mathbf{S}_j \right\rangle \tag{1.4}$$

And in the second step we consider the ansatz:

$$\langle \mathbf{S}_i \rangle \equiv M\hat{\mathbf{z}} \tag{1.5}$$

i.e. we assume that the magnetization is the same for all i. Using this we construct H^{mol}, reducing the two-body problem to a one-body problem.

In the second quantization language, we want to evaluate $\left\langle c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} \right\rangle c^\dagger_{\mathbf{k}-\mathbf{q}} c_{\mathbf{k}}$ (or other pairings, e.g. if we look at SCs we pair $c^\dagger$ with $c^\dagger$), and then posit the expectation value $\left\langle c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} \right\rangle \propto n_{\mathbf{k}} \delta_{\mathbf{q},0}$

That was pretty abstract, so let's go through the process for the concrete example of the Heisenberg magnet:

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j = -\frac{1}{2} \sum_{ij} J_{ij} \langle \mathbf{S}_i \rangle \mathbf{S}_j - \frac{1}{2} \sum_{ij} J_{ij} \mathbf{S}_i \left\langle \mathbf{S}_j \right\rangle \tag{1.6}$$

Now with the ansatz of $\langle \mathbf{S}_i \rangle = M\hat{\mathbf{z}}$ the above becomes:

$$H = -\sum_{ij} J_{ij} M S_{iz} = -\sum_i J(k=0) M S_{iz} = H^{mol} \sum_i S_{iz} \tag{1.7}$$

Now the last thing to do - we don't actually know what M is (it was a parameter in our ansatz). But, we have now reduced the Hamiltonian to a single-body Hamiltonian. So we can easily solve for $\langle S_{iz} \rangle$, and use this to (self-consistently) solve for M! Let's go through this:

$$H^{mol} \equiv J(0)M \tag{1.8}$$

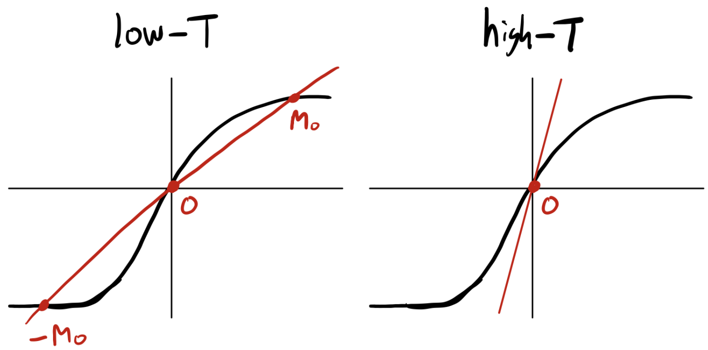
with:

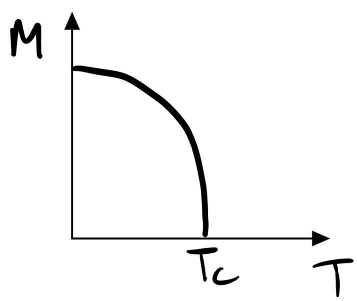
$$M = \langle S_{iz} \rangle = \frac{e^{\beta H^{mol}} - e^{-\beta H^{mol}}}{e^{\beta H^{mol}} + e^{-\beta H^{mol}}} = \tanh \beta H^{mol} \tag{1.9}$$

so we thus have the self-consistently equation:

$$M = \tanh \beta [J(0)M] \tag{1.10}$$

so we just need to look at what M solves this! This always has the trivial solution of $M = 0$. But more concretely, we can sketch the two sides and look at intersections. At low T we have three solutions $\pm M_0$ and 0 (notice the $\mathbb{Z}_2$ symmetry!) and at high T we only have the single solution of 0 so we do not have magnetization. We could plot the magnetization as a function of temperature, and would see that it is nonzero up to some T_c, above which the self-consistency equation has no solutions except for $M = 0$.





If we expand the self-consistently equation around $M = 0$, we obtain:

$$M = \beta_c J(0) M \tag{1.11}$$

which allows us to get the critical temperature:

$$k_B T_c = J(k = 0) = \sum_i J_{ij} \tag{1.12}$$

2 Mean field theory

2.1 Overview

3D superconductors are perfect mean field systems, but are difficult to think about because the order parameter is subtle; magnets are much less good MF systems, but we first study MFT in this context. Our big picture goals for today will be:

(i) When does MFT work?

(ii) Why doesn't it work for real magnets? What's missing?

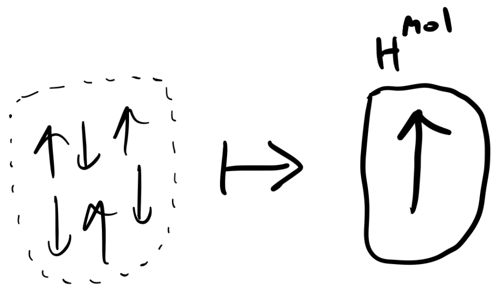
(iii) What is missing from mean field theory?

(iv) How do we improve MFT via Landau-Ginzberg Theory?

(v) What are signatures of MFT?

2.2 Reviewing/Continuing MFT for a magnet

Broadly, MFT is a tool we can use to convert an interacting many-body problem to a non-interacting one-body system in a self-consistent field. A cartoon picture is:



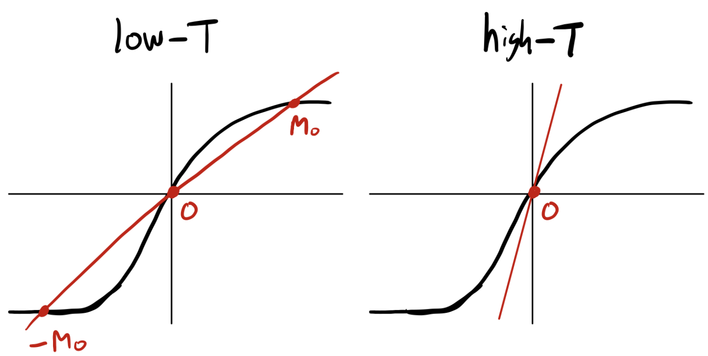
Last time, we looked at an insulating magnet:

$$\begin{aligned} H = -\frac{1}{2}\sum_{ij} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j &\overset{\text{MF step 1}}{\longrightarrow} -\frac{1}{2}\sum_{ij} J_{ij}\left\langle \mathbf{S}_i \right\rangle \cdot \mathbf{S}_j - \frac{1}{2}\sum_{ij} J_{ij}\mathbf{S}_i \cdot \left\langle \mathbf{S}_j \right\rangle \\ &\overset{\text{MF step 2}}{\longrightarrow} -\sum_{ij} J_{ij} M\hat{\mathbf{z}} \cdot \mathbf{S}_i = -\sum \mathbf{H}^{mol} \cdot \mathbf{S}_i \end{aligned} \tag{2.1}$$

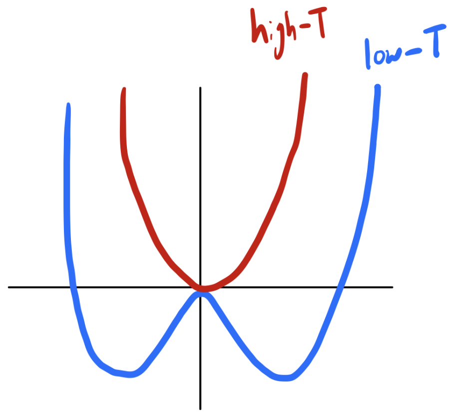
In the first step we replace $\mathbf{S}_i$ with its average, in the second step we assume there are no fluctuations/we assume that all averages of the spin are the same (and we pick it to be in the $\hat{\mathbf{z}}$ direction - though H has rotational symmetry, here we break the symmetry for MF). In SC, we will find that the average we set to be equal/have no fluctuations is of $c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}$. We then find $\langle \mathbf{S}_i \rangle$ by bootstrapping/self-consistently solving $M = \left\langle S_i^z \right\rangle$ via the definition of $H^{mol} = J(0)M = -\sum_i \mathbf{S}_i \cdot H^{mol}$:

$$M = \frac{e^{\beta H^{mol}} - e^{-\beta H^{mol}}}{e^{\beta H^{mol}} + e^{-\beta H^{mol}}} = \tanh \beta J(0) M \tag{2.2}$$

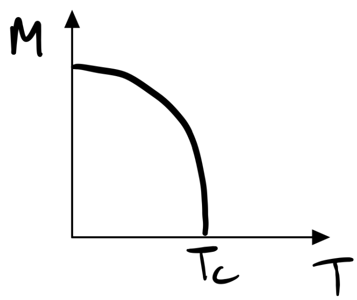
We then solve this; graphically:



looking at the Helmholtz free energies, we see a broken symmetry solution in the low-T case (with two nonzero magnetization solutions as the two minima):



Looking at minima $\frac{\partial F^{MF}}{\partial M} = 0$ we find the equilibrium magnetization:



we can expand tanh around small M to get:

$$M = \tanh(\beta_c J(0)M) \approx \beta_c J(0)M \implies \beta_c = \frac{1}{J(0)} \tag{2.3}$$

We can look at higher order terms of the expansion to see how M goes to zero/get the critical exponents:

$$M = \beta J(0)M - \frac{1}{3}[\beta J(0)M]^3 \implies M = \sqrt{3\frac{T_c - T}{T_c}} \tag{2.4}$$

Which tells us that $T \to T_c$ as $\sqrt{\frac{T_c-T}{T_c}}$, i.e. with a critical exponent of $\frac{1}{2}$.

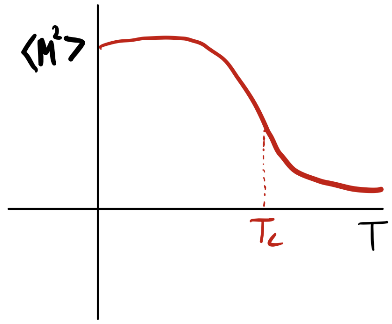
2.3 When does/doesn't MFT work?

There are 3 cases in which MFT are exact.

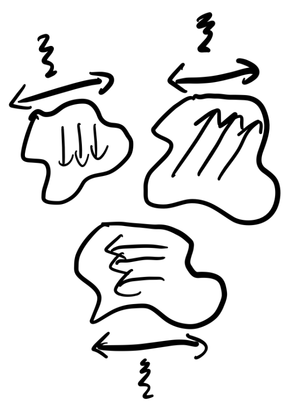
1. There an infinite number of neighbours.
2. There is an infinite range to J_{ij}.
3. There is an infinite dimension.

All of these effectively say that if all spins talk to all other spins, then we can exactly replace the interaction with an average.

Then the question becomes - when does it go wrong? For a magnet, what is missing? If we look at M in a magnet, we saw that this vanishes above T_c. But, $\left\langle M^2 \right\rangle = \langle \mathbf{M}_i \cdot \mathbf{M}_i \rangle$ does *not* vanish above T_c. It does not go to zero.



Why does this happen? we have short-range order; we have droplets of ordering (even above T_c) of size ξ (correlation length), which mean field theory ignores.



The other problem with MFT For magnets occurs near $T = 0$. The problem is the following. If I look at:

$$M(T) = \tanh(\beta T(0) M(T)) \tag{2.5}$$

there is an exponential activation $e^{-\beta J(0)}$; but this ignores a symmetry-restoring excitation. Let us make this more concrete, in the form of *Goldstone's theorem*. It says that for any finite range interaction with a continuous broken symmetry, the low-lying collective modes must be gapless - $\lim_{q\to 0} \omega_q = 0$ (compare this to the exponential activation predicted by MFT, which has a gap $\Delta = J(0)$). The reason for this is because we have lots of degeneracy of ground states arising from the continuous symmetry - there should not be any energy barrier in connecting these states! A spin wave of $q \ll 1$ (but nonzero) emerges where the spins rotate smoothly vary across space (a gentle change instead of an expensive flip predicted by MFT).

A concrete example of Goldstone bosons (though you may have not called it as such) that you have seen already are acoustic phonons. We consider a liquid-solid transition; liquids have a continuous translational symmetry vs. the discrete translational symmetry of a solid (crystal). The acoustic phonons (gapless with $\omega = c_s q$) are symmetry restoring collective modes.

2.4 Improvements to MFT: Landau-Ginzberg Theory

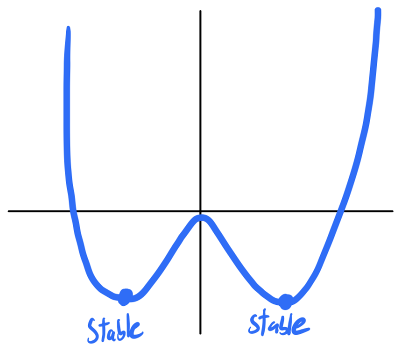
LG theory is an improvement to MFT which works for T_c (M small) for long wavelength (small q) fluctuations. The idea is that we modify/add to the MF free energy fluctuations:

$$F^{\text{LG}} = F^{\text{MF}} + F^{\text{fluc}} \tag{2.6}$$

What is the form of F^{fluc}? In it are terms $c(\nabla M)^2 = cq^2 M_q^2$. Let us back up a moment; the mean field theory piece looks like:

$$F^{\text{MF}}(M) = -AM^2 + BM^4 \tag{2.7}$$

with $A \propto \frac{T_c - T}{T_c}$ and $B \sim$ const. - this gives the Mexican hat potential that we saw earlier, with the symmetric stable solutions.



Since we want to put in the gradient term, let us rewrite this in real space:

$$F(\mathbf{r}) = -\bar{A}|M(\mathbf{r})|^2 + \bar{B}|M(\mathbf{r})|^4 + C|\nabla M(\mathbf{r})|^2 \tag{2.8}$$

The first two terms are the mean field terms (with $\bar{A} = \frac{A}{\text{Vol.}}$), and the last term is the fluctuation term, where $C \propto \xi^2$ (with ξ the coherence length). In the case where ξ is very large, and as such fluctuations become expensive; in this limit MF theory becomes more accurate as these terms dominate. In a typical superconductor, $\xi \sim 1000$ Angstroms so indeed this is the case where MF theory holds/fluctuations are suppressed.

2.5 Microscopic Derivation of Mean-Field Free Energy

We should find from our derivation that:

$$\frac{\partial F^{\text{MF}}}{\partial M} = 0 \iff M = \tanh(\beta J(0) M) \tag{2.9}$$

Physically, $F[M]$ will be a competition of energy and entropy:

$$F[M] = E[M] - TS[M] \tag{2.10}$$

energy likes large M, while the entropy likes small M, so the terms are indeed competing, with the relative strength set by temperature.

First computing S:

$$S = k_B \ln W \tag{2.11}$$

where:

$$W = \frac{N!}{N_\uparrow! N_\downarrow!} \tag{2.12}$$

Then with the constraints that:

$$N_\uparrow + N_\downarrow = N, \quad N_\uparrow - N_\downarrow = MN \tag{2.13}$$

Then:

$$N_\uparrow = \frac{N(M+1)}{2}, \quad N_\downarrow = \frac{N(1-M)}{2} \tag{2.14}$$

Now using Stirling's approximation:

$$\frac{S[M]}{N} = -k_B \left[\left(\frac{1+M}{2} \right) \ln(\frac{1+M}{2}) + \left(\frac{1-M}{2} \right) \ln(\frac{1-M}{2}) \right] \tag{2.15}$$

As $M \to 0$ we can check that $\frac{S}{N} \to k_B \ln(2)$ (high entropy) and as $M \to 1$ we can check that $\frac{S}{N} \to 0$ (low entropy/ordered).

The energy then competes with this, and we obtain it self-consistently. Let us write:

$$\mathbf{S}_i = [\mathbf{S}_i - M] + M = \mathbf{\Delta}_i + M \tag{2.16}$$

and the same for $\mathbf{S}_j$. Then:

$$H = -\frac{1}{2} \sum_{ij} J_{ij} (\Delta_i + M)(\Delta_j + M) \tag{2.17}$$

Now taking averages:

$$\langle H \rangle = - \left\langle \frac{1}{2} \sum_{ij} J_{ij} (\Delta_i + M)(\Delta_j + M) \right\rangle \tag{2.18}$$

In MF theory, we ignore the (assumed small) sproduct of deviations $\Delta_i \Delta_j$, and so:

$$\langle H \rangle_{\text{MF}} = -\frac{1}{2} J(0) M^2 = E[M] \tag{2.19}$$

So the energy and entropy go in different directions. The net effect of the algebra is that (after taking the sum of the entropic and energy terms and taking the derivative w.r.t. M):

$$\frac{\partial F^{\text{MF}}}{\partial M} = 0 \implies M = \tanh(\beta J(0) M) \tag{2.20}$$

2.6 Signatures of MFT

We will discuss some signatures of MFT - these will also be the signatures of superconductors! We have the energy:

$$E = -\frac{1}{2} J(0) M^2 \tag{2.21}$$

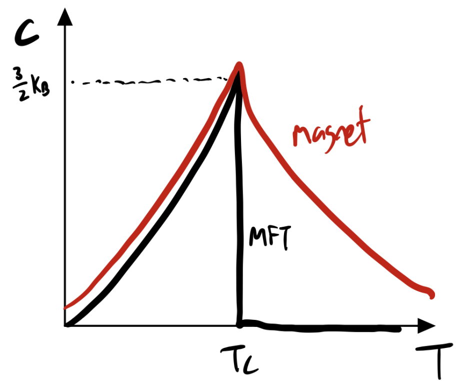
so from this we can easily define the (zero external field) specific heat (near T_c):

$$C_{H^{\text{ext}}=0} = \left. \frac{\partial E}{\partial T} \right|_{H^{\text{ext}}=0} = -MJ(0) \left(\frac{\partial M}{\partial T} \right) = -MJ(0) \frac{\partial}{\partial T} \sqrt{3 \left(\frac{T_c - T}{T} \right)} = \frac{3}{2} k_B \tag{2.22}$$

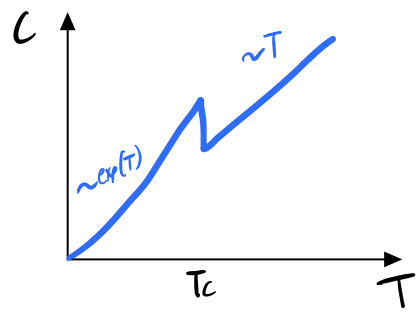
The above calculation is done below T_c; above T_c the specific heat is zero:

$$C_{H^{\text{ext}}=0} = \begin{cases} \frac{3}{2} k_B & T < T_C \\ 0 & T \geq T_c \end{cases} \tag{2.23}$$

Comparing this to, say a magnet, this step continuity is a bad prediction (it goes smoothly to zero, with a peak at the critical temperature).



But what about in a typical superconductor? Indeed there we see the MFT signature of the step discontinuity, though the specific heat does not go to zero as we still have the electron contribution of $C \sim T$ (as in a Fermi liquid).



Next class we will discuss more signatures, and then we will do a quick review of second quantization to go to the itenierate magnet.

3 Mean field theory II, Spin glasses, Second quantization

3.1 Overview of Lecture

Today we will discuss:

(i) Continued discussion of signatures of MFT

(ii) Quantification of short-range order (above T_c)

(iii) The Mermin-Wagner theorem and the 2D XY superconductivity issue[1]

(iv) Spin glasses[2]

(v) Quick introduction of second quantization

(vi) MFT of the Coloumb gas (no symmetry breaking)

As a recap, last lecture we discussed:

[1]This got the Nobel in 2016!
[2]Also a Nobel prize - in 2021 for Parisi

(i) 3D SC are ideal MFT systems (though we have not yet shown why this is true)

(ii) Magnets are not a good MFT system, due to their range of interaction being short: TODO we will study this today by computing the correlation function $\langle M(\mathbf{r})M(0)\rangle$. In particular we will find it takes the form $\frac{e^{-r/\xi_0(T)}}{r}$ (as $r \to \infty$) above T_c.

(iii) MFT only works if you have either (a) an infinite range of interaction, (b) an infinite number of nearest neighbours, or (c) an infinite dimension.

(iv) Goldstone theorem; for a continuous broken symmetry of finite range, there has to be a gapless mode, i.e. $\omega_q \to 0$ as $q \to 0$.

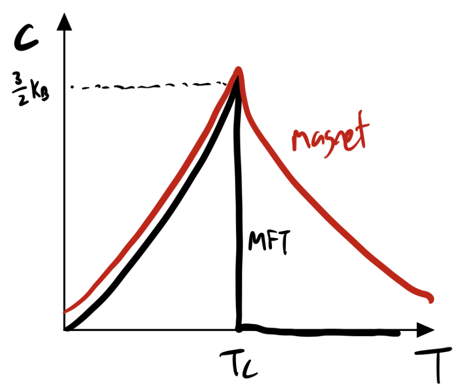
(v) Introduced the free energy functional for Landau-Ginzberg theory:

$$F^{\mathrm{LG}}[M] = F^{\mathrm{MF}}[M] + F^{\mathrm{fluc}} \tag{3.1}$$

where $F[M] = E[M] - TS[M]$, and the fluctuation term looks like:

$$F(\mathbf{r})^{\mathrm{fluc}} = |\nabla M(\mathbf{r})|^2 \to F^{\mathrm{fluc}}_{\mathbf{q}} = q^2\xi_0^2|M_{\mathbf{q}}|^2 \tag{3.2}$$

(vi) We calculated the specific heat:



and found that mean field theory ignores short range order above T_c.

3.2 Signatures of MFT

Last lecture we found a signature of MFT, namely that the C_H for magnets and C_V in superconductors shows a step discontinuity. Another signature is that of the spin susceptibility, which follows the Curie-Weiss law:

$$\chi^{\mathrm{spin}} = \lim_{H^{\mathrm{ext}}\to 0} \frac{\partial M}{\partial H^{\mathrm{ext}}} \sim \frac{1}{T - T_c} \text{ above } T_c \tag{3.3}$$

For our MFT magnet:

$$M = \tanh(\beta(J(0)M + H^{\mathrm{ext}})) \tag{3.4}$$

Computing the derivative we thus have:

$$\chi = \frac{\partial M}{\partial H} = \mathrm{sech}^2[\beta J(0)M + H]\left(\beta\left(J(0)\frac{\partial M}{\partial H} + 1\right)\right) \tag{3.5}$$

Then above T_c we have $M = 0$ for the normal state, and we thus find:

$$\chi = (1)\left(\beta J(0)\chi + 1\right) \implies \chi \sim \frac{1}{T - T_c} \tag{3.6}$$

This is a typical signature of the ferromagnetic normal state. The below T_c case is not much harder, and is left to solve on the homework.

3.3 Quantifying SRO

We want to compute the correlation function $\langle M(\mathbf{r})M(\mathbf{r}')\rangle$:

$$\left\langle M(\mathbf{r})M(\mathbf{r}')\right\rangle = \left\langle \sum_{\mathbf{k},\mathbf{k}'} M_{\mathbf{k}} E^{i\mathbf{k}\cdot\mathbf{r}} M_{\mathbf{k}'} e^{i\mathbf{k}'\cdot\mathbf{r}'} \right\rangle \tag{3.7}$$

Then by translational invariance, we know that this must be a function of $\mathbf{r} - \mathbf{r}'$, and so:

$$\left\langle M(\mathbf{r})M(\mathbf{r}')\right\rangle = \sum_{\mathbf{k}} \left\langle |M_{\mathbf{k}}|^2 \right\rangle e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \tag{3.8}$$

We can then compute $\left\langle |M_{\mathbf{k}}|^2 \right\rangle$ using Landau-Ginzberg theory:

$$F^{\text{LG}} = F^{\text{MF}} + F^{\text{fluc}} \tag{3.9}$$

where:

$$F^{LG}(\mathbf{r}) = AM(\mathbf{r})^2 + BM^4(\mathbf{r}) + C|\nabla M(\mathbf{r})|^2 \tag{3.10}$$

$$F^{LG}_{\mathbf{q}} = A|M_{\mathbf{q}}|^2 + B|M_{\mathbf{q}}|^4 + Cq^2|M_{\mathbf{q}}|^2 \tag{3.11}$$

above T_c, we can drop the B term (we just have the normal state), so all we need to do is compute a Gaussian integral:

$$\left\langle |M_{\mathbf{q}}|^2 \right\rangle = \frac{\int dm_{1k} dm_{2k} \ldots d_{m_q} |M_{\mathbf{q}}|^2 e^{-\beta F^{LG}_{\mathbf{q}}}}{\int dm_{1k} dm_{2k} \ldots d_{m_q} e^{-\beta F^{LG}_{\mathbf{q}}}} = \frac{1}{\frac{T-T_c}{T_c} + \xi_0^2 q^2} \tag{3.12}$$

This is the Ornstein-Zernicke equation. We therefore find:

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} \left\langle M(\mathbf{r})M(\mathbf{r}')\right\rangle = \lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} \int e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} |M_{\mathbf{q}}|^2 = \frac{e^{-\frac{\xi(T)}{|\mathbf{r}-\mathbf{r}'|}}}{|\mathbf{r}-\mathbf{r}'|} \tag{3.13}$$

where the correlation length is:

$$\xi(T) = \frac{\xi_0}{\sqrt{\frac{T-T_c}{T_c}}} \tag{3.14}$$

where we see the MFT exponent of $\frac{1}{2}$.

3.4 Mermin-Wagner Theorem

Formally, the MW theorem states is that long-range order is low-dimensional systems - How do we see this? We can integrate over our result for $\left\langle |M_{\mathbf{q}}|^2 \right\rangle$ in different dimensions:

$$\sum_{\mathbf{q}} |M_{\mathbf{q}}|^2 = \int \frac{d^d q}{\frac{T-T_c}{T_c} + \xi_0^2 q^2} \tag{3.15}$$

In 1D, this diverges at small q. In 2D, we have a logarithmic divergence (this is a quasi LRO, known as Kosterlitz-Thouless, very important for superconductors - the statement that superconductors are MFT systems is not strictly true in 2D. This is the 2016 Nobel, and where topology started to enter in CMT). In 3D, this converges/is well-behaved.

In PRL 17, 1133 - it is rigorously proved in 1D and 2D for isotropic Heisenberg systems that there can be no long range ferromagnetic or antiferromagnetic order (at nonzero temperature). Note that the proof relies on the Hamiltonian having a continuous symmetry, so this does not apply to the Ising model.

3.5 Spin Glasses

Spin glasses are disordered spin systems:

$$H = \sum_{ij} J_{ij}^{\text{random}} \mathbf{S}_i \cdot \mathbf{S}_j \tag{3.16}$$

for example iron spins in copper. There is $\langle \mathbf{M}_i \rangle = 0$ for all i, but, there is $\left\langle \mathbf{M}_i^2 \right\rangle \neq 0$ - the spins are "frozen in".

Let's consider a MFT for this system (note that this isn't trustworthy unless you consider the Kerpatrick model, where you send the range of the random interactions to infinity):

$$H^{\text{MF}} = \sum_{ij} J_{ij} \langle \mathbf{M}_i \rangle \cdot \mathbf{S}_j + \sum_{ij} J_{ij} \mathbf{S}_i \cdot \left\langle \mathbf{M}_j \right\rangle \tag{3.17}$$

We would then find:

$$\langle \mathbf{M}_i \rangle = \tanh(\beta \sum_{ij} J_{ij} M_j) \tag{3.18}$$

If we were to look at the free energy, we would find a free energy landscape with lots of different minima - so according to the history (T, H^{ext}), the system gets stuck in a metastable state, and the system does not escape and explore other states in any reasonable amount of time. In some sense, a glass is a kind of frozen liquid; where the system is stuck until you change the temperature and/or field. We could then see memory effects in such a system; there could be hysteresis/history dependence in the magnetization, for example.

3.6 Introduction to Second Quantization

If we recall the quantum harmonic oscillator, we had the raising and lowering operators:

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2}}(\hat{x} + i\frac{\hat{p}}{m\omega}) \\ a^\dagger &= \sqrt{\frac{m\omega}{2}}(\hat{x} - i\frac{\hat{p}}{m\omega}) \end{aligned} \tag{3.19}$$

which (using the canonical commutation relations of position and momentum) has the commutation relations:

$$[a^\dagger, a^d ag] = [a, a] = 0 \tag{3.20}$$

$$[a^\dagger_{\mathbf{q}}, a_{\mathbf{q}'}] = \delta_{\mathbf{q},\mathbf{q}'} \tag{3.21}$$

Then the first quantized Hamiltonian:

$$H^{\text{first}} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 \tag{3.22}$$

becomes in second quantization:

$$H^{\text{second}} = \omega(a^\dagger a) + \frac{1}{2} \tag{3.23}$$

or for a collection of oscillators:

$$H = \sum_{\mathbf{q}} \omega_{\mathbf{q}}(a^\dagger_{\mathbf{q}} a_{\mathbf{q}} + \frac{1}{2}) \tag{3.24}$$

and this is a generic Hamiltonian for a non-interacting Bose gas. We can also consider a fermionic versionL:

$$H^{\text{fermion}} = \sum_{\mathbf{k}} \epsilon(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{3.25}$$

with $\epsilon(\mathbf{k})$ is the energy dispersion and $c^\dagger, c$ the fermionic creation/annihilation operators:

$$\left\{ c^\dagger_{\mathbf{k}}, c_{\mathbf{k}'} \right\} = \delta_{\mathbf{k},\mathbf{k}'}. \tag{3.26}$$

We can understand $n_{\mathbf{k}\sigma} = c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$ as the number operator that counts the number of fermions with momentum $\mathbf{k}$ and spin σ. The benefit of second quantization is that we do not need to consider permutations and symmetrized/anti-symmetrized wavefunctions; we only keep track of the creation and annihilation operators, which count the number of particles for us, without all of the annoying permutation bookkeeping. The statistics are imposed through the commutation relations of the creation/annihilation operators.

So far the Hamiltonians we have wrote above are free/kinetic terms, and are fully solvable, but not very interesting. To these we can start to add interactions, e.g. the two body interaction:

which gives rise to the 2-body Hamiltonian:

$$H^{\text{2-body}} = \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} V(\mathbf{q}) c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma} c_{\mathbf{k}\sigma} c_{\mathbf{k}'\sigma} \tag{3.27}$$

If we were to do MFT with a Hamiltonian of the form $c^\dagger_1 c^\dagger_2 c_3 c_4$, we would obtain:

$$c^\dagger_1 c^\dagger_2 c_3 c_4 \to - \left\langle c^\dagger_1 c_3 \right\rangle c^\dagger_2 c_4 + \left\langle c^\dagger_1 c_4 \right\rangle c^\dagger_2 c_3 \tag{3.28}$$

4 Coloumb Gas & Hubbard Model

4.1 Overview and Review

Last day, we discussed:

- Short range order above T_c in a more quantitative way. We found that:

$$\lim_{R\to\infty} \langle M(R)M(0) \rangle \sim \frac{e^{-\frac{R}{\xi(T)}}}{R} \tag{4.1}$$

  which we can physically interepret as short-range "droplets" of aligned spin of length scale $\xi(T)$, given by:

$$\xi(T) = \frac{\xi_0}{\sqrt{\frac{T-T_c}{T_c}}} \tag{4.2}$$

  This is important for finite ξ_0, but is not important for 3D superconductors.

- The Mermin-Wagner theorem, which told us that there is no LRO in a perfect ferromagnet or anti-ferromagnet in 1-D or 2-D with complete rotational symmetry.

- 2nd quantizatio1n as a bookkeeping device. Instead of writing out a many-body wavefunctions with permutation of every particle at each site, we can just keep track of the occupation, with the exchange statistics encoded in the commutation/anticommutaitons of the creation/annihilation operators, e.g.:

$$|n_1, \ldots, n_N\rangle = (c_1^\dagger)^{n_1}(c_2^\dagger)^{n_2}\ldots(c_N^\dagger)^{n_N}|0\rangle \tag{4.3}$$

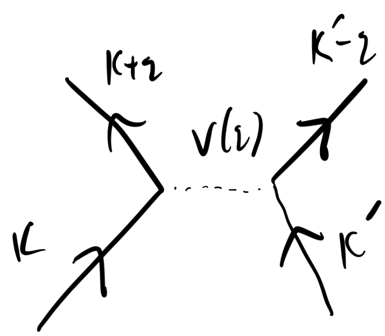
Today, we focus on a particular 2-body mean-field Hamiltonian for the repulsive interaction, without symmetry breaking (Coloumb gas) and with symmetry breaking (Hubbard Hamiltonian/itinerate ferromagnetism).

4.2 Hamiltonians in Second Quantization

Hamiltonians we write down in second quantized form look like:

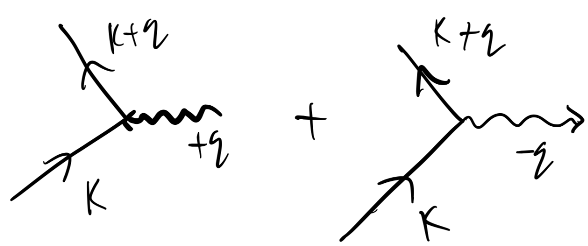
$$H = \sum_{\mathbf{k}\sigma}\epsilon(\mathbf{k})c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}\sigma\sigma'} V(\mathbf{q})c^\dagger_{\mathbf{k}+\mathbf{q}\sigma}c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'}c_{\mathbf{k}'\sigma'}c_{\mathbf{k}\sigma} \tag{4.4}$$

with $V(\mathbf{q})$ a repulsive interaction.



We can also write down the electron-phonon interaction:

$$H^{\text{el-ph}} = \sum_{\mathbf{k}\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}}c_{\mathbf{k}}(a_{\mathbf{q}} + a^\dagger_{-\mathbf{q}}) \tag{4.5}$$



Note the conservation of momentum at each vertex. We can also look at an spin, with:

$$S_{iz} = \begin{pmatrix} c^\dagger_{i\uparrow} & c^\dagger_{i\downarrow}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}c_{i\uparrow}\\ c_{i\downarrow}\end{pmatrix} \tag{4.6}$$

A spin-spin interaction (in MFT) then looks like:

$$\mathbf{S}_i\cdot\mathbf{S}_j = \left\langle c_1^\dagger c_2^\dagger c_3 c_4\right\rangle \overset{MF}{=} -\left\langle c_1^\dagger c_3\right\rangle c_2^\dagger c_4 + \left\langle c_1^\dagger c_4\right\rangle c_2^\dagger c_3 \tag{4.7}$$

where we note the minus sign that arises from exchanges of fermionic operators. There is one more pairing (of creation w/ creation) but this is what we will do on Friday when we discuss BCS, considering attractive interactions

We evaluate the expectation values to yield c-numbers:

$$\langle \psi_0 | c_1^\dagger c_3 | \psi_0 \rangle \tag{4.8}$$

and then (as we do in MFT) demonstrate self-consistency.

4.3 Solving the Coloumb gas via MFT

The mean-field Hamiltonian looks like (where we apply the averaging to the 2-body potential term)

$$H^{\text{MF}} = \sum_{\mathbf{k}} \epsilon(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}\sigma\sigma'} V(\mathbf{q}) \left\langle c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}'\sigma'} \right\rangle c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}\sigma\sigma'} V(\mathbf{q}) \left\langle c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma} \right\rangle c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} c_{\mathbf{k}'\sigma'} \tag{4.9}$$

Now, we consider evaluating the expectation values in the expression above. There is no broken symmetry - thus there is momentum conservation, in addition to the delta function of spin:

$$\left\langle c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}'\sigma'} \right\rangle = \delta_{\mathbf{q},\mathbf{k}'-\mathbf{k}} \delta_{\sigma\sigma'} \langle n_{\mathbf{k}'\sigma'} \rangle \tag{4.10}$$

This is the exchange (or Hartree) term, and we also have the direct (or Fock) term:

$$\left\langle c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma} \right\rangle = \delta_{\mathbf{q},\mathbf{0}} \langle n_{\mathbf{k}\sigma} \rangle \tag{4.11}$$

Thus:

$$H^{\text{MF}} = \sum_{\mathbf{k}} \epsilon(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} V(\mathbf{k}'-\mathbf{k}) \delta_{\sigma\sigma'} \langle n_{\mathbf{k}'\sigma'} \rangle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} V(\mathbf{0}) \langle n_{\mathbf{k}'\sigma} \rangle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{4.12}$$

Which corresponds to a shift in the kinetic energy:

$$\epsilon(\mathbf{k}) \to \tilde{\epsilon}(\mathbf{k}) - \sum_{\mathbf{k}'\sigma} V(\mathbf{k}'-\mathbf{k}) \langle n_{\mathbf{k}'\sigma'} \rangle \delta_{\sigma\sigma'} + \sum_{\mathbf{k}'\sigma'} V(\mathbf{0}) \langle n_{\mathbf{k}'\sigma} \rangle . \tag{4.13}$$

The occupations are given by the Fermi-dirac distributions:

$$\langle n_{\mathbf{k}\sigma} \rangle_{\text{perturbation}} = \frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\mu)}+1} \tag{4.14}$$

but if we do Hartree-Fock self-consistently, we replace $\epsilon(\mathbf{k})$ with $\tilde{\epsilon}(\mathbf{k})$:

$$\langle n_{\mathbf{k}\sigma} \rangle_{\text{self-consistent}} = \frac{1}{e^{\beta(\tilde{\epsilon}_{\mathbf{k}}-\mu)}+1} \tag{4.15}$$

Now evaluating these contributions, the exchange term is found to be:

$$V = \frac{4\pi e^2}{|\mathbf{k}-\mathbf{k}'|^2} \tag{4.16}$$

and the direct term is neutral and can be set $V(\mathbf{0}) = 0$. Thus:

$$\tilde{e}_k = \epsilon_k + \int d^3k' \frac{f(\epsilon_{\mathbf{k}'})}{|\mathbf{k}-\mathbf{k}'|^2} \tag{4.17}$$

Of course this is highly divergent, but is repaired by screening:

$$\frac{V(\mathbf{q})}{\epsilon(\mathbf{q},0)} = \frac{\frac{4\pi e^2}{q^2}}{1+\frac{q_{TF}^2}{q^2}} = \frac{4\pi e^2}{q^2+q_{TF}^2}. \tag{4.18}$$

4.4 Hubbard Model

Hubbard truncated the screened Coloumb Hamiltonian (with $\sim V(r) \sim \frac{e^{-\lambda r}}{r}$) to give rise to the Hubbard Hamiltonian:

$$H_{\text{Hubbard}} = \sum \epsilon(\mathbf{k}) c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{i\sigma} U n_{i\sigma} n_{i-\sigma}. \tag{4.19}$$

Note that only spins with opposite spins talk to each other! We can now solve this in MFT, replacing:

$$U n_{i\sigma} n_{i-\sigma} \rightarrow U \langle n_{i\sigma} \rangle n_{i-\sigma} \tag{4.20}$$

Giving rise to the MF Hamiltonian:

$$H^{\text{MF}}_{\text{Hubbard}} = \sum_{\sigma} \left[\epsilon(\mathbf{k}) + U \langle n_{i-\sigma} \rangle \right] c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{4.21}$$

With:

$$\langle \tilde{n}_{i-\sigma} \rangle = \frac{1}{N} \sum_{\mathbf{k}} \frac{1}{e^{\beta(\tilde{\epsilon}(\mathbf{k}) - \mu)} + 1} \tag{4.22}$$

note that the ϵ appearing in the above expectation value is replaced with the $\tilde{\epsilon}$ (the new/renormalized kinetic energy) - it is boostrapped so that we solve self-consistently. We can now see if magnetism emerges.

Now taking $n_{\uparrow} + n_{\downarrow} = n$, $n_{\uparrow} - n_{\downarrow} = m$, with m a variational parameter, and $n_{\uparrow}, n_{\downarrow}$ the up/down spin populations:

$$n_{\uparrow} = \frac{n+m}{2}, \quad n_{\downarrow} = \frac{n-m}{2} \tag{4.23}$$

We then find from self-consistency that:

$$m = \frac{1}{N} \sum_{\mathbf{k}} \left[\frac{1}{e^{\beta(\epsilon_{\mathbf{k}} + u(\frac{n-m}{2}) - \mu)} + 1} + \frac{1}{e^{\beta(\epsilon_{\mathbf{k}} + u(\frac{n+m}{2}) - \mu)} + 1} \right] \tag{4.24}$$

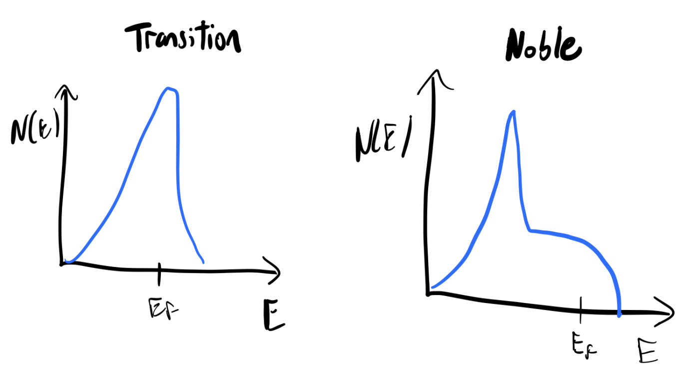
Then for small m:

$$m = \frac{um}{2} \frac{\partial n}{\partial \mu} + \frac{um}{2} \frac{\partial n}{\partial \mu} = um \frac{\partial n}{\partial \mu} \tag{4.25}$$

So then we find that for $1 < u \dfrac{\partial n}{\partial \mu}$ that we are unstable to magnetism, with:

$$\frac{\partial n}{\partial \mu} \sim N(E_F) \sim \frac{KE}{E_F} \tag{4.26}$$

So the system is competing with the potential energy wanting aligned spins (ferromagnetism) and the kinetic energy (no ferromagnetism). This theory is used to explain Fe, Ni, Co (transition metals) which are band magnets. Conversely, noble metals (Au, Ag, Cu) do not see the same density of states at the fermi energy and have different behaviour.



There is one more thing that Hubbard is good for, namely for very strong U this is a good model for antiferromagnetism, with virtual hopping between anti-aligned spin sites. This leads to it being a hot topic in high-T_c superconductivity.

4.5 Primer for BCS - Bogoliubov Transformation

If we instead have an attractive potential:

$$\sum V_{\mathbf{k}\mathbf{k}'} c_1^\dagger c_2^\dagger c_3 c_4 \to \sum V \left\langle c_1^\dagger c_2^\dagger \right\rangle c_3 c_4 + \text{h.c.} \tag{4.27}$$

This no longer looks like the kinetic energy term (which has a number operator), as we now have two annihilation/creation terms together. To remedy this, we consider transforming to a new set of variables:

$$\begin{aligned} \alpha &= uc^\dagger + vc \\ \alpha^\dagger &= uc - vc^\dagger \end{aligned} \tag{4.28}$$

We can then invert this to write $c/c^\dagger$ in terms of $\alpha, \alpha^\dagger$. Then, the transformed Hamiltonian looks like:

$$H^{\text{new}} = A\alpha^\dagger\alpha + B\alpha^\dagger\alpha^\dagger + B^*\alpha\alpha + E_0 \tag{4.29}$$

and then we choose u, v such that $B = B^* = 0$ and only the number operator term survives, we thus get that that new Hamiltonian is in a new diagonal term with operators that correspond to neither particles nor hole, but a new type of particles - they are a new type of quasi-particle that keep the fermionic statistics which they are linear combinations of.

Note that Bogoliubov mostly did the bosonic problem, but the formalism works for either case.

5 BCS Theory

5.1 Review + Overview

Lat time, we looked at repulsive many-body interactions, in particular studying the Coloumb gas:

$$H^{\text{Coulomb}} = \sum_{\mathbf{k}\sigma} \epsilon(\mathbf{k}) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum V(\mathbf{q}) c_{\mathbf{k}+\mathbf{q}\sigma}^\dagger c_{\mathbf{k}'-\mathbf{q}\sigma'}^\dagger c_{\mathbf{k}\sigma} c_{\mathbf{k}'\sigma'} \tag{5.1}$$

which we could study the mean-field version of by replacing the two-body interaction with the average $\left\langle c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}'\sigma'} \right\rangle \propto \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$, i.e. no symmetry breaking. We also studied the Hubbard model (which leads to itinerate magnetism and antiferromagnetism). The latter is what we get when we truncate the Coloumb interaction to short range, obtaining:

$$H^{\text{Hubbard}} = \sum_{\mathbf{k}\sigma} \epsilon(\mathbf{k}) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{5.2}$$

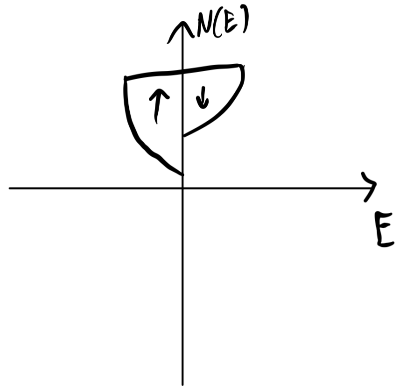
Which in the MF limit becomes:

$$H^{\text{Hubbard}} = \sum_{\mathbf{k}\sigma} \epsilon(\mathbf{k}) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + U \sum_i \left\langle n_{i\uparrow} \right\rangle n_{i\downarrow} + U \sum_i n_{i\uparrow} \left\langle n_{i\downarrow} \right\rangle \tag{5.3}$$

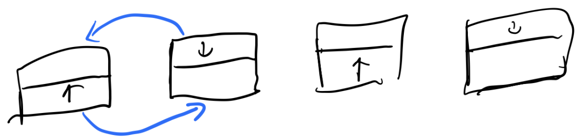
Then we look for a self-consistent solution:

$$\langle n_{i\sigma} \rangle = \langle n_\sigma \rangle = \frac{1}{N} \sum \frac{1}{e^{\beta(\epsilon_{\mathbf{k}\sigma} - \mu)} + 1} \tag{5.4}$$

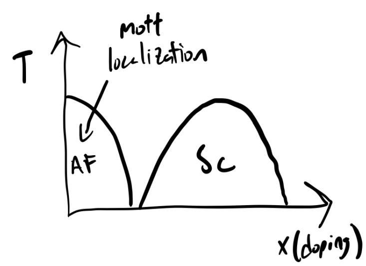
which admits a solution with $\left\langle n_\uparrow \right\rangle \neq \left\langle n_\downarrow \right\rangle$, leading to band ferromagnetism with a density of states that looks like:



We also discussed, briefly, how a strong U turns this model into an antiferromagnet. This may be relevant to understanding superconductivity in the cuprates. If we look at real space around 1/2 filling, i.e. where each site has one particle in it, with alternating spin up/down. This gives rise to an exchange interaction $J_{AFM}^{\text{exchange}}$, which is related to superconductivity (actually, magnetic order is related to it more generally).



If we look at Cuprate phase transitions, e.g. $La_{2-x}Sr_xCuO_4$ or $YBaCuO$, we see:



If we turn the crank and do the Hartree-Fock variational method on the attractive story, we would obtain the results of the BCS theory (just by pairing of terms), but we would not understand how they came about. So, let's work towards the understanding; to this end, we will need to understand two-fluid physics.

5.2 Ideal Bose Gas & Two-fluid physics

To start, we will look at the ideal Bose gas. The task is to solve the number equation for bosons:

$$N = \sum_{\mathbf{k}} \frac{1}{e^{\beta(\epsilon_{\mathbf{k}} - \mu)} - 1} \tag{5.5}$$

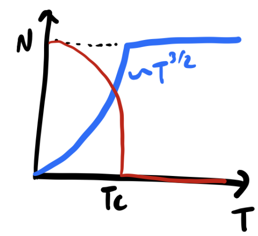
for $\mu(T)$. $\mathbf{k} = \mathbf{0}$ is problematic already, but this is where the condensate will live; let N_0 be the number of bosons in this state:

$$N_0 = \frac{1}{e^{-\beta\mu} - 1} \tag{5.6}$$

If $\mu > 0$, then $\frac{1}{e^{-|\beta\mu|}-1}$ and so $N_0 < 0$ which is unphysical. If $\mu > 0$, then we have $\frac{1}{e^{|\beta\mu|}-1}$ which is too small as $\beta \to \infty$ to satisfy the equation. Thus, $\mu = 0$ is the only option at low temperature. But if $\mu = 0$, then our above expression for N_0 is ill-defined. So let's take a slightly different approach. We have $N_0 = N - N_{\text{ex}}$ (where N_{ex} are all of the $\mathbf{k} \neq 0$ states) then:

$$N_{\text{ex}} = \sum_{\mathbf{k}} \frac{1}{e^{\beta\epsilon_{\mathbf{k}}} - 1} = \int g(\epsilon) \frac{d\epsilon}{e^{\beta\epsilon} - 1} \sim T^{3/2} \tag{5.7}$$

So then we find, plotting N_{ex} and N_0:



So we can understand the two populations (the condensate/ground state fluid at $\mathbf{k} = 0$ and the excited fluid for $\mathbf{k} \neq 0$) as the "two fluids". We can calculate other thermodynamics in the system, such as the energy:

$$E = \int \epsilon g(\epsilon) d\epsilon \frac{1}{e^{\beta\epsilon} - 1} \tag{5.8}$$

Which tuggests $E \sim T^{5/2}$ and $c_V \sim T^{3/2}$. We find T_c as the temperature when $N^{\text{ex}} = N$. Below this T_c, we have a macroscopic occupation of $N_0 = N_{\mathbf{k}=0}$, i.e. the condensate emerges. The $c_V \sim T^{3/2}$ implies that we have a q^2 dispersion for excitations.

How does this ideal Bose gas compare with Helium-4?

	Ideal	He-4
T_c (K)	3.14	2.2
$N_{\mathbf{k}=0}$ at $T = 0$	100%	10%
c_V	$T^{3/2}$	T^3
Excitation Dispersion	$\omega \sim q^2$	$\omega = c_s q$ (sound modes)

Comparing BCS with a Bose Gas, in BCS we have attractive interactions, gapped excitations, and 100% condensation at $T = 0$. A weakly interacting Bose gas is instead repulsive, has gapless excitation, and a few percent are out of the condensate.

We now turn the crank, but keep an eye out for what the physics is doing as we do so.

5.3 First stab at BCS

We study:

$$H - \mu N = \sum_{\mathbf{k}} (\epsilon(\mathbf{k}) - \mu) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}\sigma\sigma'} V_{\mathbf{k}\mathbf{k}'\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} c_{\mathbf{k}\sigma} c_{\mathbf{k}'\sigma'} \tag{5.9}$$

We then pair $c^\dagger_{\mathbf{k}+\mathbf{q}\sigma}, c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'}$, with:

$$\left\langle c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} \right\rangle \sim \delta_{\mathbf{k}',-\mathbf{k}} \delta_{\sigma',-\sigma} \tag{5.10}$$

We pick pairs with $\pm\mathbf{k}$ (zero momentum) and $\pm\sigma$ (singlet pairing).

We define $\Delta_\mathbf{k}$:

$$\Delta_\mathbf{k} = -\sum V_{\mathbf{k}\mathbf{k}'} \left\langle c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} \right\rangle \tag{5.11}$$

So then the mean field Hamiltonian becomes:

$$H^{MF} - \mu N = \sum_{\mathbf{k}\sigma} \xi(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum \Delta_\mathbf{k} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + h.c. \tag{5.12}$$

where $\Delta_\mathbf{k}$ becomes our self-consistent parameter. We now carry out the Bogoliobov transformation to obtain fermionic quasiparticle creation/annihilation operators (corresponding to the fermionic condensation):

$$\begin{aligned} \alpha_\mathbf{k} &= u_\mathbf{k} c_{\mathbf{k}\uparrow} - v_\mathbf{k} c^\dagger_{-\mathbf{k}\downarrow} \\ \alpha^\dagger_{-\mathbf{k}\downarrow} &= u_\mathbf{k} c^\dagger_{-\mathbf{k}\downarrow} - v_\mathbf{k} c^\dagger_{\mathbf{k}\uparrow} \end{aligned} \tag{5.13}$$

with $u_\mathbf{k}^2 + v_\mathbf{k}^2 = 1$ to imporse fermionic statistics. Now, inverting this we have:

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= u_\mathbf{k} \alpha_{\mathbf{k}\uparrow} + v_\mathbf{k} \alpha^\dagger_{-\mathbf{k}\downarrow} \\ c^\dagger_{-\mathbf{k}\downarrow} &= u_\mathbf{k} \alpha^\dagger_{-\mathbf{k}\downarrow} - v_\mathbf{k} \alpha_{\mathbf{k}\uparrow} \end{aligned} \tag{5.14}$$

from which we get:

$$A\alpha^\dagger\alpha + B\alpha\alpha + B^*\alpha^\dagger\alpha^\dagger + E_0 \tag{5.15}$$

which we can set $B = 0$ to diagonalize the Hamiltonian in terms of the quasiparticles. More precisely, we end up with:

$$H^{\text{MF}} - \mu N = E_0 + \sum_{\mathbf{k}\sigma} \left[\xi(\mathbf{k}(u_\mathbf{k}^2 - v_\mathbf{k}^2) + 2\Delta_\mathbf{k} u_\mathbf{k} v_\mathbf{k})\right] \alpha^\dagger_{\mathbf{k}\sigma}\alpha_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\sigma} [2\xi(\mathbf{k}) u_\mathbf{k} v_\mathbf{k} + \Delta_\mathbf{k}(v_\mathbf{k}^2 - u_\mathbf{k}^2)] \alpha_{\mathbf{k}\sigma}\alpha_{\mathbf{k}\sigma} + \text{h.c.} \tag{5.16}$$

We want to set the $\alpha\alpha$ term to zero. From the $u_\mathbf{k}^2 + v_\mathbf{k}^2 = 1$ condition we find that $u_\mathbf{k} = \sin\theta_\mathbf{k}$ and $v_\mathbf{k} = \cos\theta_\mathbf{k}$, and so $\cos 2\theta_\mathbf{k} = v_\mathbf{k}^2 - u_\mathbf{k}^2$ and $\sin 2\theta_\mathbf{k} = 2u_\mathbf{k} v_\mathbf{k}$. We thus obtain the equation:

$$\xi(\mathbf{k}) \sin 2\theta_\mathbf{k} + \Delta_\mathbf{k} \cos 2\theta_\mathbf{k} = 0 \implies \tan 2\theta_\mathbf{k} = -\frac{\Delta_\mathbf{k}}{\xi(\mathbf{k})} \tag{5.17}$$

So then:

$$\begin{aligned} \cos 2\theta_\mathbf{k} &= \frac{-\xi_\mathbf{k}}{\sqrt{\xi_\mathbf{k}^2 + \Delta_\mathbf{k}^2}} \\ \sin 2\theta_\mathbf{k} &= \frac{\Delta_\mathbf{k}}{\sqrt{\xi_\mathbf{k}^2 + \Delta_\mathbf{k}^2}} \end{aligned} \tag{5.18}$$

From which we find:

$$E_\mathbf{k} = \left[\xi(\mathbf{k}(u_\mathbf{k}^2 - v_\mathbf{k}^2) + 2\Delta_\mathbf{k} u_\mathbf{k} v_\mathbf{k})\right] = \frac{\Delta_\mathbf{k}^2 + \xi_\mathbf{k}^2}{\sqrt{\Delta_\mathbf{k}^2 + \xi_\mathbf{k}^2}} = \sqrt{\Delta_\mathbf{k}^2 + \xi_\mathbf{k}^2} \tag{5.19}$$

So our conclusion:

$$H^{\mathrm{MF}} - \mu N = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \alpha^{\dagger}_{\mathbf{k}\sigma} \alpha_{\mathbf{k}\sigma} \tag{5.20}$$

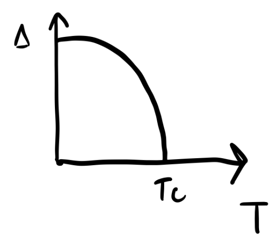
with:

$$E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta_{\mathbf{k}}^2}. \tag{5.21}$$

If we look at the Fermi function:

$$f(E) = \frac{1}{e^{\beta\sqrt{(\epsilon-\mu)^2+\Delta^2}} + 1} \tag{5.22}$$

which goes to zero as $\beta \to \infty$. We also see the presence of an excitation/energy gap. What does this mean, physically? Δ must be the "bound state energy", i.e. the energy needed to break (Cooper) pairs in the ground state condensate. This Δ we will see will act as an order parameter:



This we have not shown yet, but we will show it via the self-consistency condition.

5.4 The Superconducting gap equation

The loose end we have to tie up is verifying when:

$$\Delta_{\mathbf{k}} = -\sum V_{\mathbf{k}\mathbf{k}'} \left\langle c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} \right\rangle_{\mathrm{mf}} \tag{5.23}$$

is a consistent definition. We now enforce this constraint, which we call the gap equation. We will do thius vy substituting in αs for the cs above, and then computing the expectation value in terms of the αs.

$$\Delta_{\mathbf{k}} = -\sum V_{\mathbf{k}\mathbf{k}'} \left\langle (u_{\mathbf{k}'} \alpha^{\dagger}_{\mathbf{k}\uparrow} + v_{\mathbf{k}'} \alpha_{-\mathbf{k}'\downarrow})(u_{\mathbf{k}'} \alpha^{\dagger}_{-\mathbf{k}'\downarrow} - v_{\mathbf{k}'} \alpha^{\dagger}_{\mathbf{k}'\uparrow}) \right\rangle_{\mathrm{mf}} \tag{5.24}$$

Now $\langle \alpha\alpha \rangle = \left\langle \alpha^{\dagger}\alpha^{\dagger} \right\rangle = 0$ and then $\alpha\alpha^{\dagger} = 1 - \alpha^{\dagger}\alpha$, so we just end up with two of the pairings:

$$\Delta_{\mathbf{k}} = -\sum V_{\mathbf{k}\mathbf{k}'} (-1) u_{\mathbf{k}'} v_{\mathbf{k}'} f(E_{\mathbf{k}'}) - \sum V_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} (1 - f(E_{\mathbf{k}'})) \tag{5.25}$$

So then from $u_{\mathbf{k}} v_{\mathbf{k}} = \frac{\sin 2\theta_{\mathbf{k}}}{2} = \frac{\Delta_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}}$, we obtain:

$$\Delta_{\mathbf{k}} = -\frac{1}{2} \sum_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{\sqrt{\xi_{\mathbf{k}'}^2 + \Delta_{\mathbf{k}'}^2}} (1 - 2f(E_{\mathbf{k}'})) \tag{5.26}$$

wich is the BCS gap equation, which allows us to solve (at any temperature - temperature enters through the Fermi function) for the superconducting gap.

5.5 Teaser - Superconducting ground state

A question is what does the superconducting ground state look like? It looks like (proposed by Schrieffer):

$$|\psi_0\rangle = \prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|\psi_0\rangle \tag{5.27}$$

and what is left to prove that $\alpha_{\mathbf{k}}|\psi_0\rangle = 0$ to see that this is indeed the GS.

Historical aside - BCS paper fatal error discovered by Nambu, in particular they broke gauge invariance of the theory via violating Ward identities - Nambu patched this up. But, still a beautiful paper worth reading.

6 BCS Theory II

6.1 Overview and Revier

This week we will continue discussing BCS theory, and then we will talk about superfluidity in cold atom systems.

Last time, we looked at extended Hartree theory, where we considered $\left\langle c^\dagger_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow}\right\rangle \neq 0$, which corresponded to fermionic excitations aboe a (currently unknown) ground state of bound fermion pairs. We obtained the diagonalized mean-field theory Hamiltonian:

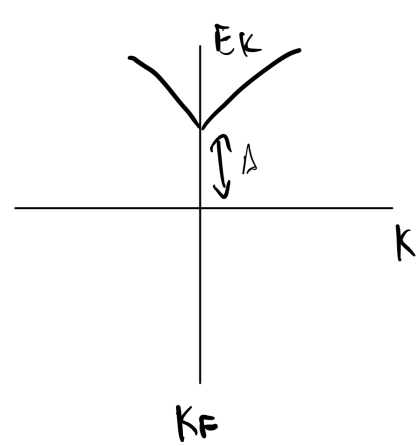
$$H^{\text{mf}} = E_0 + \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}\sigma} \alpha_{\mathbf{k}\sigma} \tag{6.1}$$

with $E_k = \sqrt{\Delta^2_{\mathbf{k}} + \xi^2_{\mathbf{k}}}$ with $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ and the fermi function:

$$f(E_{\mathbf{k}}) = \frac{1}{e^{\beta E_{\mathbf{k}}} + 1} \to 0 \tag{6.2}$$

as $T \to 0$. We found the self-consistency equation for the superconducting gap, which is the energy requried for excitations:

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left\langle c^\dagger_{\mathbf{k}'\uparrow} c_{-\mathbf{k}'\downarrow}\right\rangle = -\frac{1}{2}\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{\sqrt{\xi^2_{\mathbf{k}'} + \Delta^2_{\mathbf{k}'}}}(1 - 2f(E_{\mathbf{k}'})) \tag{6.3}$$



6.2 Toy Model for BCS Potential

Assume a toy model (attractive) for $V_{\mathbf{k}\mathbf{k}'}$. This will turn out to be related to photons since $T_c \sim \frac{1}{\sqrt{M}}$ (which gives an isotope effect) and involve the Debye energy ω_D. In particular:

$$V_{\mathbf{k}\mathbf{k}'} = -V\Theta(\omega_D - |\xi_{\mathbf{k}}|)\theta(\omega_D - |\epsilon_{\mathbf{k}'}|) \tag{6.4}$$

with Θ the step function. The gap is then:

$$\Delta_{\mathbf{k}} = \Delta\Theta(\omega_D - |\xi_{\mathbf{k}}|) = \frac{1}{2}\sum V \frac{\Theta(\omega_D - |\xi_{\mathbf{k}'}|)}{\sqrt{\xi_{\mathbf{k}'}^2 + \Delta_{\mathbf{k}'}^2}} \Delta(\omega_D - |\xi_{\mathbf{k}}|)\Delta \tag{6.5}$$

Thus cancelling terms on both sides:

$$\Delta_{\mathbf{k}} = \frac{1}{2} V \int_{-\omega_D}^{\omega_D} d\xi_{\mathbf{k}'} \frac{\Delta_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}'}^2 + \Delta_{\mathbf{k}}^2}} N(E_F) \tag{6.6}$$

which is the BCS gap equation for the toy model. We can solve this numerically, but we can also consider two analytically limits; when $T \to T_c$, then $\Delta \to 0$ and $E_{\mathbf{k}} = |\xi_{\mathbf{k}}|$ and $T \to 0$ wherein $\Delta \to \Delta_0$. The results one gets in these limits are:

$$\frac{2\Delta(T=0)}{T_c} = 3.5 \tag{6.7}$$

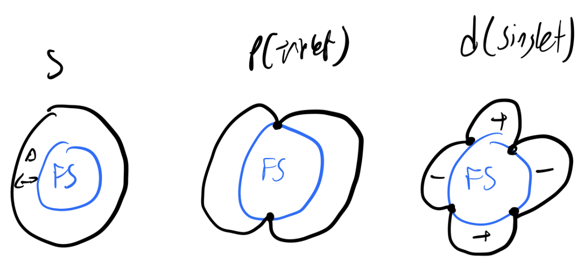
$$k_B T_C \sim 1.14\omega_D e^{-\frac{1}{N(E_F)V}} \sim \frac{1}{\sqrt{M}} (\text{isotope effect}) \tag{6.8}$$

Where this second result gives the isotope effect result, as well as a non-analytic dependnece of T_c on the coupling constant. Experimentally, these can be measured:

V	Zn	Al	Ga	Nb	Hg	Pb
3.4	3.2	3.3	3.5	3.8	4.6	4.3

The last two entries are strong coupling superconductors so the toy model doesn't predict as well. Also note that this story doesn't really apply to high-T_c superconductors, as there are no isotope effect in the cuprates.

Note that $\Delta_{\mathbf{k}}$ has k-dependence, and we can have s-wave ($T_c \leq 10 - 20$K), p-wave (He-3 superfluids), d-wave superconductors ($T_c \sim 100$K, strong (molecular) pairs)...



6.3 BCS Ground State Wavefunction

Schrieffer determined the form of the BCS wavefunction. We believe that this form of the ground state holds even with very strong pairing.

$$|\psi_0\rangle = \prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle \tag{6.9}$$

with:

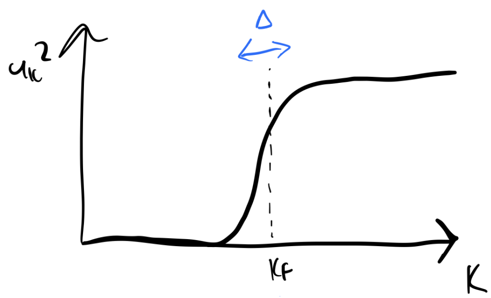
$$\begin{aligned} u_{\mathbf{k}} &= \sin\theta_{\mathbf{k}} \\ v_{\mathbf{k}} &= \cos\theta_{\mathbf{k}} \\ u^2_{\mathbf{k}} + v^2_{\mathbf{k}} &= 1 \\ v^2_{\mathbf{k}} - u^2_{\mathbf{k}} &= \cos 2\theta_{\mathbf{k}} = -\frac{\xi_{\mathbf{k}}}{E} \end{aligned} \tag{6.10}$$

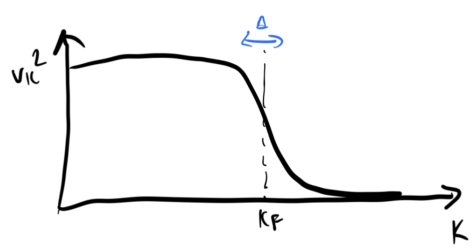
Thus:

$$\begin{aligned} u^2_{\mathbf{k}} &= \frac{1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}}{2} \\ v^2_{\mathbf{k}} &= \frac{1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}}{2} \end{aligned} \tag{6.11}$$

The interpretation of the BCS wavefunction is that the first term corresponds to empty pairs and the second term corresponds to occupied pairs. Thus we rearrange the Fermi sea where we have $\mathbf{k}, -\mathbf{k}$ pairs.

Let's look at the graphical behaviour of the coefficients. As $\xi \to -\infty$, $u^2_{\mathbf{k}} \to 0$ and as $\xi \to +\infty$ $u^2_{\mathbf{k}} \to 1$, and sketching we have:





For $\Delta \ll E_F$, we thus have a slightly reorganized Fermi sea, getting a "blurring" of the edge of the Fermi surface, with smeared occupation. At the blurring, we have particle-hole mixing $\alpha^\dagger = c^\dagger + c$. In principle, there is not too much different from a regular fluid though - The number of electrons that contribute to superconductivity are on the order of $N \sim \frac{\Delta}{E_F}$. You can use this to estimate the condensation energy gain (how much energy does the system gain via the attractive interaction and not being a fermi liquid and goes superconducting) and it turns out to be $\frac{\Delta^2}{E_F}$. But if $\Delta \sim 10$K and $E_F \sim 10^4$K, the energy gain is only:

$$\frac{\Delta^2}{E_F} \sim 10^{-2}\text{K} \tag{6.12}$$

the system does it, but it is small.

People at one point thought you could keep increasing the electron-phonon coupling to increase T_c, but it turns out when increasing too far the system just changes the lattice to reduce the energy, and we no longer have a superconductor.

6.4 Proof of correctness of the BCS wavefunction

We can show that it indeed it is the ground state by showing that the state has no excitations:

$$\alpha_{-\mathbf{p}\downarrow}|\psi_0\rangle \stackrel{?}{=} 0 \tag{6.13}$$

First observe that:

$$\alpha_{-\mathbf{p}\downarrow} = (u_{\mathbf{p}}c_{-\mathbf{p}\downarrow} + v_{\mathbf{p}}c^\dagger_{\mathbf{p}\uparrow}) \tag{6.14}$$

$$\begin{aligned}
\alpha_{-\mathbf{p}\downarrow}|\psi_0\rangle &= \alpha_{-\mathbf{p}\downarrow}\prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow})|0\rangle \\
&= (u_{\mathbf{p}}c_{-\mathbf{p}\downarrow} + v_{\mathbf{p}}c^\dagger_{\mathbf{p}\uparrow})(u_{\mathbf{p}} + v_{\mathbf{p}}c^\dagger_{\mathbf{p}\uparrow}c^\dagger_{-\mathbf{p}\downarrow})\prod_{\mathbf{k}\neq\mathbf{p}}(u_{\mathbf{k}} + v_{\mathbf{k}}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow})|0\rangle \\
&= -u_{\mathbf{p}}v_{\mathbf{p}}c^\dagger_{\mathbf{p}\uparrow}(\prod\ldots)|0\rangle + u_{\mathbf{p}}v_{\mathbf{p}}c^\dagger_{\mathbf{p}\uparrow}(\prod\ldots)|0\rangle \\
&= 0
\end{aligned} \tag{6.15}$$

Where we use that the $c_{-\mathbf{p}\downarrow}|0\rangle$ term vanishes, and that $(c^\dagger_{\mathbf{p}})^2 = 0$ vanishes (fermions) as well. We also use that $c_{-\mathbf{p}\downarrow}c^\dagger_{\mathbf{p}\uparrow}c^\dagger_{-\mathbf{p}\downarrow} = -c^\dagger_{\mathbf{p}\uparrow}$ when acting on the ground state. This doesn't "prove" it is the ground state, but it is consistent with it being the ground state.

What if we look at quasi-particle excitations on top of the ground state? Then:

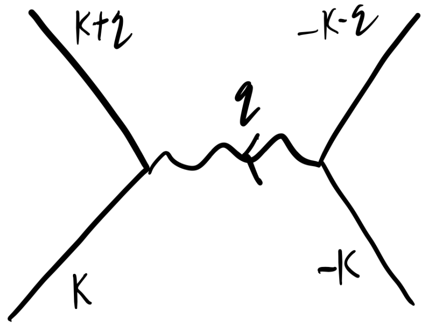
$$\begin{aligned}
\alpha^\dagger_{\mathbf{p}\uparrow}|\psi_0\rangle &= (c^\dagger_{\mathbf{p}\uparrow}u_{\mathbf{p}} - v_{\mathbf{p}}c_{-\mathbf{p}\downarrow})(u_{\mathbf{p}} + v_{\mathbf{p}}c^\dagger_{\mathbf{p}\uparrow}c^\dagger_{-\mathbf{p}\downarrow})\prod_{\mathbf{k}\neq\mathbf{p}}\ldots|0\rangle \\
&= (u^2_{\mathbf{p}} + v^2_{\mathbf{p}})c^\dagger_{\mathbf{p}\uparrow}\prod_{\mathbf{k}\neq\mathbf{p}}\ldots|0\rangle
\end{aligned} \tag{6.16}$$

so we see that the excitation "breaks" a pair in the ground state and releases a bare fermion as an excitation.

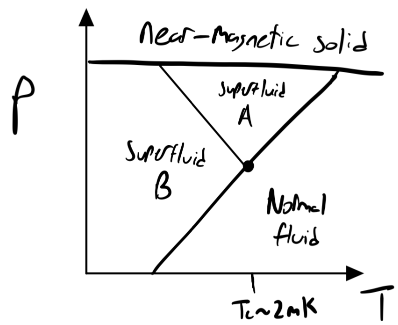
6.5 The "glue" of superconductivity

How do phonons cause attraction between fermions? This is not the only game in town, because ultracold atoms, He-3, cuprates etc. this is not the mechanism, but let's get the intuition for common superconductors. The phonon attraction is time-retarded - we have positive charges, and an electron comes in, modifying the positive charge, modifying it (a kind of screening cloud) before quickly leaving. Then another electrons come in, the positive ion distortion from the first electron remains (ions relax slowly), and the second electron is attracted to this.

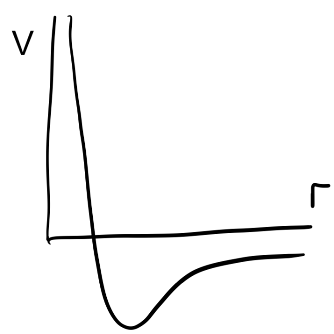
Drawing this process as a Feynman diagram (really to see the concrete phonon interaction you would have to solve this consistently through field theory):



How does the story differ in superfluid He-3?



People always thought there would be a superfluid in He-3, due to the Van der Waals interaction which was argued to win out at low temperatures. Theorists kept refining their guesses/lowering the predicted T_c. The superfluid was found by accident; people were probing the solid-FM phase and then found little blips/signals corresponding to superfluidity. Perhaps the fact that the phase was FM means that it liked odd ℓ pairing, which corresponds to p-waves.



Another different story is in Alkali Fermi-gases. How do we get BCS-BEC behaviour? Where is the attraction? It's related to the Feshback resonance - a magnetic field dials in a strong attraction. We will also study alkali Bose gases (weakly interacting Bose gaes). Interestingly, these require repulsion; else they collapse (Bosenova). In this context, the Hamiltonian looks like:

$$H = \sum \mathbf{k}\epsilon(\mathbf{k}) b^\dagger_{\mathbf{k}} b_{\underline{\mathbf{k}}} + \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'} V^{\text{rep}}(\mathbf{q}) b^\dagger_{\mathbf{k}+\mathbf{q}} b^\dagger_{\mathbf{k}'-\mathbf{q}} b_{\mathbf{k}'} b_{\mathbf{k}}. \tag{6.17}$$

The main experiemtal effort on this front was lowering the temperature low enough to see it while keeping the system gaseous; this required $T_c \sim 2^{\text{-}}$K for both the fermions and the bosons.

Though we will look at different systems, but the mathematical machinery we have constructed will be the same!

Next class we will look at the variational/free energy approach to BCS, and then the Bogolioubov story (where the pairing will be a little bit strange).

7 BCS Theory III

7.1 Overview + Review

Today, we will be a little more qualitative, going over superconducting phenomena and classifying them. Next week we will go over dilute Bose gases and see how the Bogolioubov theory looks like in that context.

Last time, we discussed the BCS ground state:

$$|\psi_0\rangle = \prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle \tag{7.1}$$

with the first term the empty pairs and the second term the occupied pairs. Last time we sketched what $u_{\mathbf{k}}^2, v_{\mathbf{k}}^2$ looked like, with $u_{\mathbf{k}}^2$ staying at zero until near k_F wherein it grows (with characteristic range Δ) to 1 (and $v_{\mathbf{k}}^2$ follows the opposite trend). If $\Delta \ll E_F$, this is a very small distortion of the Fermi surface; at the surface we have a mixing of particles and holes (this is how we can physically interpret the quasiparticle operators $\alpha^\dagger = uc^\dagger - vc$). For consistency, we saw that $\alpha_{-\mathbf{p}\downarrow}|\psi_0\rangle = 0$, as is expected for the ground state with no excitations. We could also see excitations above the ground state:

$$\alpha^\dagger_{\mathbf{p}\downarrow}|\psi_0\rangle = c^\dagger_{\mathbf{p}\uparrow}\prod_{\mathbf{k}\neq\mathbf{p}}(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle \tag{7.2}$$

where we saw a free fermion emerge. The number of pairs is *not* conserved. But we could see this just from the ground state, where:

$$\langle N|N+2\rangle = \langle\psi_0|c^\dagger c^\dagger|\psi_0\rangle = 0 \tag{7.3}$$

In some sense the N is no longer a good quantum number. We can also consider the expectation value:

$$\left\langle \hat{N} \right\rangle = \langle\psi_0|\sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}|\psi_0\rangle = \langle c_{\mathbf{k}\sigma}\psi_0|c_{\mathbf{k}\sigma}\psi_0\rangle = \sum_{\mathbf{k}\sigma} v_{\mathbf{k}}^2 = \sum_{\mathbf{k}\sigma}\left(\frac{1-\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}}{2}\right) = \left(\frac{1}{2}+\frac{1}{2}\right)N = N \tag{7.4}$$

where in the second-to-last equality we use that $\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} = 0$. The takeaway - the expectation value is indeed N, but the ground state is not an eigenstate of the number operator.

7.2 Characteristics of Superconductors

There are largely two classes of experiments:

(I) Things that measure the condensate. In this context, we can use Landau-Ginzberg theory (emphasis on order parameter Δ). We will see:

$$F \sim A|\Delta|^2 + B|\Delta|^4 + C|\boldsymbol{\nabla}\Delta|^2 \tag{7.5}$$

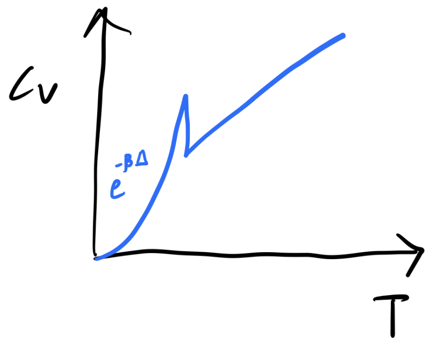
where the gradient term is where the vector potential comes in.

(II) Things that measure the excitations/fermions. In this class, we can use BCS theory (emphasis on fermions). More difficult in the presence of currents (have to recalculate everything, have to be careful about Gauge invariance) vs. easier to add an extra term into free energy.

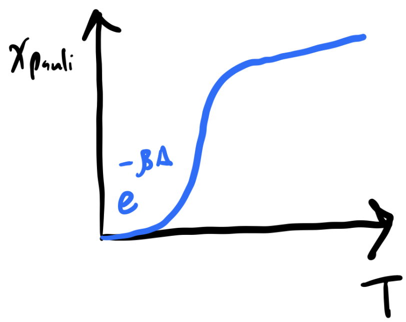
So, what are some experimental characteristics?

1. Evidence for excitation gap

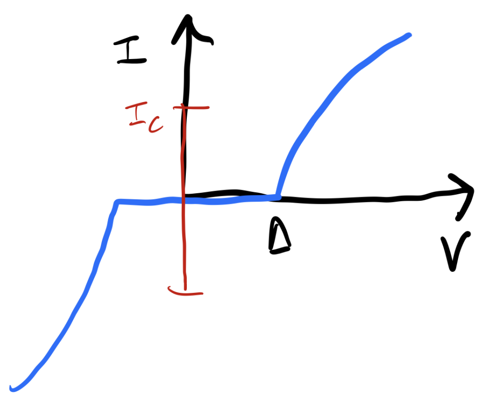
    - C_V at low T (fermions)



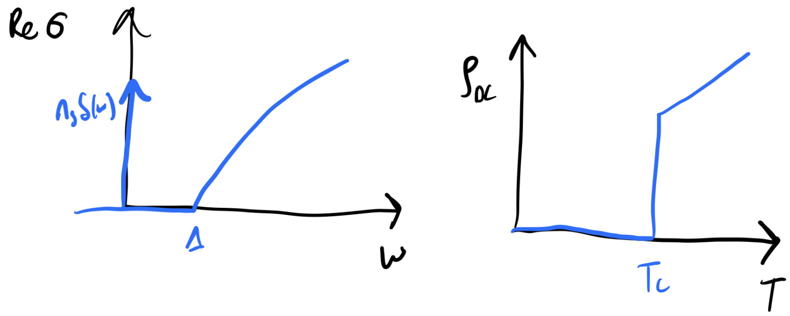
    - χ_{Pauli} are low T (fermions). Sketched is for singlet (triplet has nonzero χ at $T = 0$).



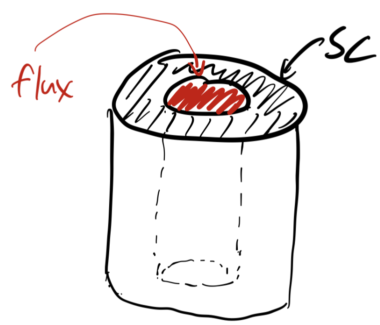
    - $I - V$ curve (blue is normal metal next to superconductor (fermions), red is Josephson junction with two superconductors next to each other, pair tunneling (condensate)). In the former (at $T = 0$), I can't get any current until I apply voltage $> \Delta$. In the latter, the IV curve looks the same but we do have the $V = 0$ blip.



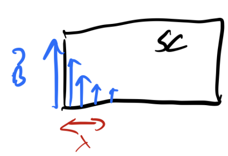
    - Conductivity/Resistivity. There's a gap piece (fermions) and there's a zero-frequency piece (hard to measure) coming from the (condensate). Below T_c we have no resistivity, and this is coming from the zero-frequency delta function; the bosons are responsible for this sudden jump.



2. Flux quantization (condensate) - in multiply connected superconductors, we have quantized trapped flux of:
$$\Phi_q = \frac{hc}{q} = \frac{hc}{2e} \tag{7.6}$$
And this comes from the condensate as the condensate has $2e$ charge pairs (as opposed to the bare fermion excitations).



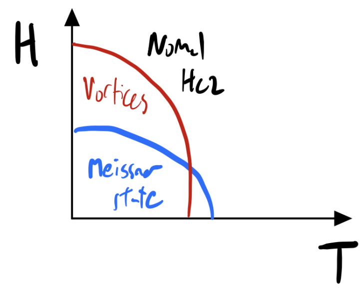
3. Flux expulsion in the bulk; there is a small (distance) magnetic field penetration, with characteristic scale/London penetration depth λ. This happens because the condensate can cheaply set up screening currents.



4. Proximity effects; put a superconductor next to a normal metal (say Pb). The superconductivity does not stay confined and leaks into the neighbouring material. This idea has been extended to topological superconductors, where placing a topological insulator next to a metal may give rise to a topological superconductor.

We will go through each of these (using the tools of LG/BCS).

Also aside - there are type-II superconductors; invented by Abrikosov. We have phase diagram:



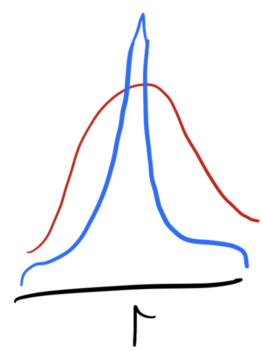
Where we have the formation of vortices/droplets, with normal fluid inside of them. The H can penetrate through these vortices.

7.3 Superfluidity in alkali atomic gases

We have Bose condensates and Fermi condensates. These are neutral, so we don't have a lot of the regular probes. We instead measure (indirectly) via density profiles.We have gases in a trapping potential (initial experiments in 1995 by Wieman and Cornell):

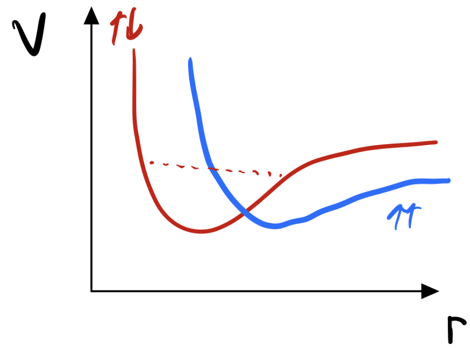
$$V(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \tag{7.7}$$

Above $T_c \sim 2$¯K,, we have a broad peak at the center, and below T_c we get a sharp delta peak (with thermal tails).

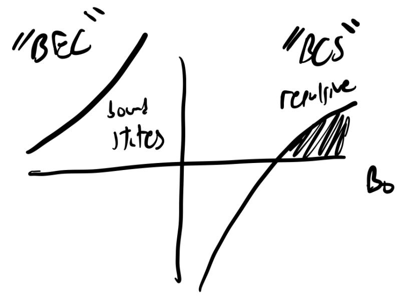


The smoking gun that superfluidity was happening was through the observation of quantized vortices through laser spoons. They even have a kind of Josephson effect.

Fermions are even harder because cooling is hard (Pauli exclusion). Underlying the approach here is the idea of Fechbach resonance. We can tune the two-particle interaction in H. We have two competing scattering states in the Van der Waals potential, one triplet and one singlet.



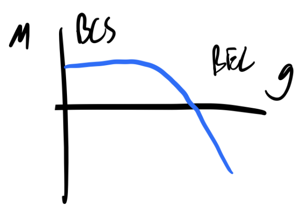
We get a scattering resonance when the bound state lines up with the triplet channel. We can tune this via a magnetic field (moving up the triplet curve - singlet channel does not move because there is no net spin). We thus can tune between short ξ/small Cooper pairs vs. large ξ/large Cooper pairs.



How to understand this? If we take the BCS wavefunction;

$$|\psi_0\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle \tag{7.8}$$

and self-consistently solve for the chemical potential μ, we find that past a certain coupling/interaction strength g, the chemical potential becomes negative, so we lose the Fermi surface; we then in this regime effectively have bosons and get BEC.



Comment about pseudogaps; Δ_{sc} goes to zero at T_c but pairing gap can persist, hence a "pseudogap". There is a T^*, which is a large temperature below which there is an onset of pairing. When g is large, although T_c may drop, T^* simply keeps growing. Why does T_c get small then? $m^* \to 0$ so we can no longer have hopping (as the paired fermions cannot hop until they unglue). So it's not actually a good superconductor despite very strong "glue".

Next week we'll discuss the dilute Bose gas, discuss the details behind the phenomonelogy, and discuss variational approach to BCS.

8 Weakly Interacting Bose Gases

8.1 Overview + Review

Friday - topics list for papers/talks.

Today we will discuss Bose gases, which turn out to be more difficult to treat than Fermi gases. Note that this theory does not describe He-4 well, but does describe well trapped alkali gases. In this setting we have $q = 0$ bosons in the condensate and $q \neq 0$ bosons (phonons).

Last lecture we discussed different classes of experiments, some which discuss the fermionic excitations (described by BCS) and some that describe the Bose condensate (Landau-Ginzberg). Surely, these two sets of tools could not be independent.

8.2 Connection between LG and BCS

The basis for Landau-Ginzberg theory is a thermodynamic potential:

$$\Omega = E - TS - \mu N \tag{8.1}$$

wherein near T_c, Ω satisfies variational conditions - in the simplest case:

$$\Omega = A\Delta^2 + B\Delta^4, \quad \frac{\partial \Omega}{\partial \Delta} = 0 \tag{8.2}$$

where the above condition should yield the same result as the BCS gap equations. If we parameterize $\Omega(\theta_k, \xi_k)$ and variationally solve, we should find:

$$\frac{\partial \Omega}{\partial f} = 0 \implies f = f_k = \frac{1}{e^{\beta E_k} + 1} \tag{8.3}$$

$$\frac{\partial \Omega}{\partial \Theta} \implies \tan\theta_k = \frac{-\xi_k}{\Delta_k} \tag{8.4}$$

What we have is:

$$\Omega^{\text{BCS}} = E_{\text{mf}} - \mu N - TS \tag{8.5}$$

with:

$$-TS = -2k_B T \sum_k f_k \ln f_k + (1 - f_k)\ln(1 - f_k) \tag{8.6}$$

how do we get $E_{\text{mf}} - \mu N$? What we did in class was:

$$E_{\text{mf}} - \mu N = \left\langle H^{\text{mf}} \right\rangle = \left\langle \sum_k E_k \alpha^\dagger_{k\sigma} \alpha_{k\sigma} + E_0 \right\rangle + \text{const.} \tag{8.7}$$

The only issue is that in class, we dropped the constant term (and you will need it for the HW). This constant is a function $g(\theta_k, f_k)$. What we did in class was:

$$V c^\dagger c^\dagger c c \to V \left\langle c^\dagger c^\dagger \right\rangle c c + \text{h.c.} \tag{8.8}$$

Let us then write:

$$c^\dagger c^\dagger = [c^\dagger c^\dagger - \left\langle c^\dagger c^\dagger \right\rangle] + \left\langle c^\dagger c^\dagger \right\rangle \tag{8.9}$$

$$cc = [cc - \langle cc \rangle] + \langle cc \rangle \tag{8.10}$$

So when we take the average (with $\delta = c^\dagger c^\dagger - \left\langle c^\dagger c^\dagger \right\rangle, \delta' = cc - \langle cc \rangle$ small)

$$\left\langle V[\delta + \left\langle c^\dagger c^\dagger \right\rangle][\delta' + \langle cc \rangle] \right\rangle \implies \text{const.} = V \left\langle c^\dagger c^\dagger \right\rangle \langle cc \rangle \tag{8.11}$$

which we can then evaluate the averages in terms of the αs. We then get the same equations as we did from the BCS derivation from the variational conditions.

A slight preview of LG theory. We have the thermodynamic potential:

$$\Omega^{\text{LG}} = -A|\Delta|^2 + B|\Delta|^4 + C|\boldsymbol{\nabla}\Delta|^2 \to -A|\Delta|^2 + B|\Delta|^4 + C|(\boldsymbol{\nabla} - \frac{2eA}{c})\Delta|^2 + \frac{H^2}{8\pi} \tag{8.12}$$

The vector potential $\mathbf{A}(r)$ and the gap $\Delta(r)$ are deeply connected - we can solve for these self-consistently via a variation $\Delta \to \Delta + \delta\Delta$ and $A \to A + \delta A$. Therein we have the current $J \sim \psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^*$ with $\psi \sim \Delta$.

8.3 Weakly Interacting Bose Gas - Wavefunction

We will see a superfluid emerge from this theory. It is in between an ideal Bose gas (no interactions) and He-4 (strong interactions).

The Hamiltonian looks almost like a BCS Hamiltonian:

$$H = \sum_{\mathbf{k}} \epsilon(\mathbf{k}) b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} V^{\text{repulsive}} b^\dagger_{\mathbf{k}+\mathbf{q}} b^\dagger_{\mathbf{k}'-\mathbf{q}} b_{\mathbf{k}'} b_{\mathbf{k}} \tag{8.13}$$

Bosons or Cooper pairs in BCS are essentially non-interacting; we have wavefunctions:

$$|\psi_0\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}) |0\rangle \tag{8.14}$$

where the second term looks like a boson:

$$b_0^\dagger = c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \tag{8.15}$$

So it looks like we are populating the ground state with spinless, zero-momentum bosons.

We now consider the Bogolioubov wavefunction:

$$|\psi^{\text{Bog}}\rangle = e^{[b_0^\dagger N_0 - \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} b^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}}]} |0\rangle \tag{8.16}$$

which we can compare to the BCS wavefunction:

$$|\psi^{\text{BCS}}\rangle = e^{\sum_{\mathbf{k}} \frac{u_{\mathbf{k}}}{v_{\mathbf{k}}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}} |0\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}) |0\rangle \tag{8.17}$$

8.4 Phenomenology

In a bose condensate, we have the two-fluid system. The normal fluid has finite viscosity and the superfluid has zero viscosity.

If we put He-4 into a dewer, superfluid leaks out of every pore; it cannot be contained. There is also a travelling up the walls and leaking out due to Van-der Waals. If we rotate the dewer, the superfluid does not rotate (due to the zero viscosity). If I were to stir the superfluid with a laser spoon, I create vortices/holes inside the superfluid. They have quantized vorticity, and persist in time. This is somewhat analogous to the magnetic-field-in type-II superconductor story, where there are vortices emerging; there is an analog between $\mathbf{F} = \mathbf{v} \times \mathbf{B}$ (Lorentz) and $\mathbf{F} = \mathbf{v} \times \boldsymbol{\omega}$ (Coriolis).

Note that we do not see the strong repulsion we see in He-4 in trapped alkali gases. We have two scattering channels, triplet and singlet, both with bound states. You can turn the external/applied field to get a bound state resonance, which corresponds to a divergence in the scattering/interaction strength. So we might ask why can't we just work in a regime where the interaction is very strong and simulate He-4 that way. But, in this setting we work in the regime where there are bound states. Away from the scattering divergence, we can have either a stable Bose condensate (for $B \gg 1$) or a stable molecular condensate (for $B \ll -1$). But near the scattering divergence we have that the states collapse/are unstable.

Stable molecular condensates are generically quite difficult to prepare. We have two atoms in our condensate... but all the internal degrees of freedom need to be removed/we can't, e.g., have rotational excitations. And such internal excitations are much more numerous. At UChicago, Cheng Chin, Dave Demille, and Zoe Yan are all working on such molecular experiments.

8.5 Solving the Theory

Back to the Hamiltonian:

$$H = \sum_{\mathbf{k}} \epsilon(\mathbf{k}) b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} V^{\text{rep}} b^\dagger_{\mathbf{k}+\mathbf{q}} b^\dagger_{\mathbf{k}'-\mathbf{q}} b_{\mathbf{k}'} b_{\mathbf{k}} \tag{8.18}$$

We looked at the mean-field theory for the spin chain and for BCS, which involved different kinds of averaging/pairing. How we proceed here is by assuming that most bosons are in the ground state/there are very few excited bosons (this is the small T limit - but unphysical things happen when we take the temperature up, so this theory only holds in this limit). We assume $\langle b_{\mathbf{k}} \rangle \neq 0$ (but small) and take $\langle b_{\mathbf{k}=0} \rangle = b_0 \equiv \sqrt{N_0}$. The leading order term is that every particle is in the condensate:

$$\langle b_0 \rangle^4 \sim N_0^2 \tag{8.19}$$

The next term is where we pick $\mathbf{k}, \mathbf{k}', \mathbf{q}$ such that two bs are b_0 and the other are not. We have six such terms:

(i) $\mathbf{k}, \mathbf{k}' = 0 \implies \frac{1}{2} V(\mathbf{q}) b^\dagger_{\mathbf{q}} b^\dagger_{\mathbf{q}} b_0 b_0$ (pair creation)

(ii) $\mathbf{k} = -\mathbf{q}, \mathbf{k}' = \mathbf{q} \implies \frac{1}{2} V(\mathbf{q}) b^\dagger_0 b^\dagger_0 b_{\mathbf{k}} b_{-\mathbf{k}}$ (pair annihilation)

(iii) $\mathbf{q} = 0, \mathbf{k} = 0 \implies \frac{1}{2} V(0) b^\dagger_0 b^\dagger_{\mathbf{k}'} b_{\mathbf{k}'} b_0$ (scattering (Hartree))

(iv) ...

(v) ...

(vi) ...

So when the smoke clears, the mean-field Hamiltonian is:

$$\begin{aligned} H^{\text{mf}} = &\frac{1}{2} \sum_{\mathbf{k} \neq 0} \epsilon(\mathbf{k}) (b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + b^\dagger_{-\mathbf{k}} b_{-\mathbf{k}}) + \frac{1}{2} \sum_{\mathbf{q}} N_0 V(\mathbf{q}) [b^\dagger_{\mathbf{q}} b_{\mathbf{q}} + b^\dagger_{-\mathbf{q}} b_{-\mathbf{q}}] \\ &+ \frac{1}{2} \sum_{\mathbf{k} \neq 0} N_0 V(\mathbf{q}) [b^\dagger_{\mathbf{q}} b^\dagger_{-\mathbf{q}} + b_{\mathbf{q}} b_{-\mathbf{q}}] + \frac{1}{2} V(0) N_0^2 + \frac{N_0}{2} \sum V(0) (b^\dagger_{\mathbf{q}} b_{\mathbf{q}} + b^\dagger_{-\mathbf{q}} b_{-\mathbf{q}}) \end{aligned} \tag{8.20}$$

which we can diagonalize via Bogolioubov transformation:

$$H^{\text{mf}} = \sum_{\mathbf{k}} E_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + E_0 \tag{8.21}$$

With the self-consistency equation:

$$N_0 \approx N \tag{8.22}$$

i.e. a "small depletion" of the condensate at $T = 0$. We'll go through more of the algebra next class, but we can talk about some of the research going on with bose gases to end of f the lecture.

8.6 Equilibriation (and lack thereof) of Bose Gases

To describe equilibriation, we can do time-dependent Landau-Ginzberg theory, also known as Gross-Pitaevskii theory. Experimentally, we could consider:

- Optical lattices - boson kinetic energies obey Bloch's theorem $E(k) \sim \sin(ka)$. If we have an upper and lower band, with a condensate originally in the lower band, we can use a Raman pulse to kick it up to the upper band and then watch the condensate roll down. We could consider a sinusoidal trap $V(r, \omega t)$ in time, leading to a new kind of band structure (Floquet bands). Quenches - we can make a quick variation of the scattering length $A(t)$ (Feschbach setting). By doing this sweep/quench, we can make a unitary/metastable Bose gas, with persistent oscillations.

9 Bose Superfluids

Today we will discuss Bose superfluids and tunnelling in superconductors.

See paper topics list in canvas. You will want to give a pedagogical overview. Be sure to include material beyond textbooks, i.e. recent papers (published within the last 10 years). Outline + bibliography due Feb 19.

9.1 Possible Paper Topics

- Iron based superconductors (Fe-Se). T_c in monolayer films, 60K or so. Usually called iron pnictides.
- Color superconductivity
- 2D superconductivity Kosteviltz-Thouless. There is no real long-range order in 2D, but there is short-range SC, topological defects, vortices etc.
- Fullerides - Buckeyball carbon with alkali
- Strontium Ruthanate (triplets)
- Atomic gases superfluids alkali atoms (includes BCS-BEC)
- Heavy fermions
- Organic superconductors
- Superfluid He-3 (magnetism induced superconductivity)
- Supersolids
- Quantum criticality + superconductivity
- Topological SC, majorana particles
- Moire superconductors
- Hydride superconductivity
- Pair density wave superconductors
- SQUIDs
- Particle Physics (Higgs)
- Exotic Magnetism/Spin glasses
- SC in Floquet/driven systems

9.2 Review of Weakly Interacting Bose Gas

Last lecture, we discussed the weakly interacting Bose condensate:

$$H = \sum_{\mathbf{k}} \epsilon(\mathbf{k}) b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V(\mathbf{q}) b^\dagger_{\mathbf{k}+\mathbf{q}} b^\dagger_{\mathbf{k}'-\mathbf{q}} b_{\mathbf{k}'} b_{\mathbf{k}} \tag{9.1}$$

We assume that $\left\langle b^\dagger_{\mathbf{k}} \right\rangle \neq 0$ and set $\left\langle b^\dagger_{\mathbf{k}=0} \right\rangle \equiv b_0$. We used mean-field theory to replace the four-body term with:

$$b_0 b_0 b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + b_0 b_0 + b^\dagger_{-\mathbf{k}} b_{-\mathbf{k}} \tag{9.2}$$

with $b_0 = \sqrt{N_0}$. This requires two essential assumptions. First that we are close to zero temperature, $T \approx 0$, and that V is small, namely that the bosons are weakly interacting. The V is responsible for kicking particles out of the condensate, so we require it to be small for $N_0 \approx N$. The mean field Hamiltonian looked like:

$$H^{\text{mf}}_{\text{boson}} = \frac{1}{2} \sum_{\mathbf{q}} \xi_{\mathbf{q}} (b^\dagger_{\mathbf{q}} b_{\mathbf{q}} + b^\dagger_{-\mathbf{q}} b_{-\mathbf{q}}) + \frac{1}{2} \sum_{\mathbf{q}} \Delta_{\mathbf{q}} (b^\dagger_{\mathbf{q}} b^\dagger_{-\mathbf{q}} + b_{\mathbf{q}} b_{-\mathbf{q}}) + V(0)^{\text{mf}} \tag{9.3}$$

with:

$$\xi_{\mathbf{q}} = \epsilon(\mathbf{q}) + N_0 V(\mathbf{q}) \tag{9.4}$$

$$\Delta_{\mathbf{q}} = N_0 V(\mathbf{q}) \tag{9.5}$$

This looks like BCS, but we have different definitions of ξ, Δ and we have commutation relations:

$$[b^\dagger_{\mathbf{q}}, b_{\mathbf{q}'}] = \delta_{\mathbf{q}\mathbf{q}'} \tag{9.6}$$

The picture is that of a condensate with very few excitations.

We consdier the Bogolioubov transformation:

$$\begin{aligned} \alpha_{\mathbf{q}} &= u_{\mathbf{q}} b_{\mathbf{q}} + v_{\mathbf{q}} b^\dagger_{-\mathbf{q}} \\ \alpha^\dagger_{-\mathbf{q}} &= u_{\mathbf{q}} b^\dagger_{-\mathbf{q}} + v_{\mathbf{q}} b_{\mathbf{q}} \end{aligned} \tag{9.7}$$

with $u^2_{\mathbf{q}} - v^2_{\mathbf{q}} = 1$. We can invert this to obtain the bs in terms of the αs, which gives:

$$H^{\text{mf}} = \sum E_{\mathbf{q}} \alpha^\dagger_{\mathbf{q}} \alpha_{\mathbf{q}} \tag{9.8}$$

with:

$$E_{\mathbf{q}} = \sqrt{\xi^2_{\mathbf{q}} - \Delta^2_{\mathbf{q}}} \tag{9.9}$$

now the excitations are gapless! The only self-consistency equation we have here to close the loop is to take $N_0 \approx N$.

9.3 Bose Superfluid Dispersion

we parametrize $u_{\mathbf{q}} = \cosh\theta_{\mathbf{q}}$ and $v_{\mathbf{q}} = \sinh\theta_{\mathbf{q}}$, then after substituting in the expressions for the αs we have:

$$H^{\text{mf}} = (\ldots)\alpha^\dagger\alpha + (\ldots)[\alpha^\dagger\alpha^\dagger + \alpha\alpha] + (\ldots)\text{const} \tag{9.10}$$

with $\alpha^\dagger\alpha$ having the coefficient:

$$\frac{1}{2} \sum_{\mathbf{q}} \xi_{\mathbf{q}} (u^2_{\mathbf{q}} + v^2_{\mathbf{q}}) - 2u_{\mathbf{q}} v_{\mathbf{q}} \Delta_{\mathbf{q}} \tag{9.11}$$

and the off diagonals having the coefficient:

$$\frac{1}{2} \sum_{\mathbf{q}} \xi_{\mathbf{q}} (-2u_{\mathbf{q}} v_{\mathbf{q}}) + \Delta_{\mathbf{q}} (u^2_{\mathbf{q}} + v^2_{\mathbf{q}}) \tag{9.12}$$

which we set to vanish, yielding the conditions:

$$u_{\mathbf{q}}^2 + v_{\mathbf{q}}^2 = \cosh 2\theta_{\mathbf{q}} \tag{9.13}$$

$$2u_{\mathbf{q}}v_{\mathbf{q}} = \sinh 2\theta_{\mathbf{q}} \tag{9.14}$$

which yields:

$$\tanh 2\theta_{\mathbf{q}} = \frac{\Delta_{\mathbf{q}}}{\xi_{\mathbf{q}}} \tag{9.15}$$

We then end up with the diagonalized Hamiltonian:

$$H^{\text{mf}} = \frac{1}{2}\sum_{\mathbf{q}} E_{\mathbf{q}}\alpha_{\mathbf{q}}^{\dagger}\alpha_{\mathbf{q}} + \frac{1}{2}\sum_{\mathbf{q}} E_{-\mathbf{q}}\alpha_{-\mathbf{q}}^{\dagger}\alpha_{-\mathbf{q}} + V^{\text{mf}}(0) \tag{9.16}$$

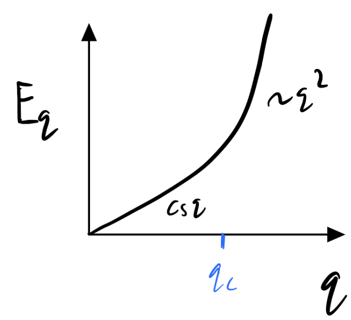
with:

$$E_{\mathbf{q}} = \sqrt{\xi_{\mathbf{q}}^2 - \Delta_{\mathbf{q}}^2} = \sqrt{[\epsilon(\mathbf{q}) + N_0 V(\mathbf{q})]^2 - (N_0 V(\mathbf{q}))^2} \tag{9.17}$$

For small $\mathbf{q}$:

$$E_{\mathbf{q}} \approx \sqrt{\epsilon(\mathbf{q}) 2N_0 V(0)} \to c_s|\mathbf{q}| \tag{9.18}$$

if we sketch the dispersion:



where we see the sound-mode like behaviour for small $\mathbf{q}$ and for large $\mathbf{q}$ we recover the free particle dispersion.

9.4 Consistency condition

To prove consistency, we require:

$$N - N_0 = \left\langle \sum_{\mathbf{q}} b_{\mathbf{q}}^{\dagger} b_{\mathbf{q}} \right\rangle = N^{\text{ex}} \tag{9.19}$$

we evaluate this sum by replacing the bs with the αs:

$$N^{\text{ex}} = \sum_{\mathbf{q}} \left\langle (u_{\mathbf{q}}\alpha_{\mathbf{q}}^{\dagger} - v_{\mathbf{q}}\alpha_{-\mathbf{q}})(u_{\mathbf{q}}\alpha_{\mathbf{q}} - v_{\mathbf{q}}\alpha_{-\mathbf{q}}^{\dagger}) \right\rangle \tag{9.20}$$

now, $\langle \alpha\alpha \rangle = \left\langle \alpha^{\dagger}\alpha^{\dagger} \right\rangle = 0$ and $\left\langle \alpha^{\dagger}\alpha \right\rangle = 0$ for $T = 0$ (set it to zero for leading order), so we are just left with:

$$N^{\text{ex}} = \sum_{\mathbf{q}} v_{\mathbf{q}}^2 \tag{9.21}$$

Now, since we have:

$$u_{\mathbf{q}}^2 + v_{\mathbf{q}}^2 = \frac{\xi_{\mathbf{q}}}{E_{\mathbf{q}}} \tag{9.22}$$

and

$$u_{\mathbf{q}}^2 - v_{\mathbf{q}}^2 = 1 \tag{9.23}$$

We find:

$$v_{\mathbf{q}}^2 = \frac{\frac{\xi_{\mathbf{q}}}{E_{\mathbf{q}}} - 1}{2} \tag{9.24}$$

Which is a complicated integral, but from it we extract;

$$\sum_{\mathbf{q}} v_{\mathbf{q}}^2 \propto |V(0)|^{3/2} \tag{9.25}$$

which tells us that the number out of the condensate is small if $V(0)$ is small, i.e. not a lot of repulsion.

There is one important length scale, namely:

$$q_c \sim \frac{1}{\xi} \sim \frac{1}{2mc_s} \tag{9.26}$$

which we will need when we look at the LG theory for the bosons.

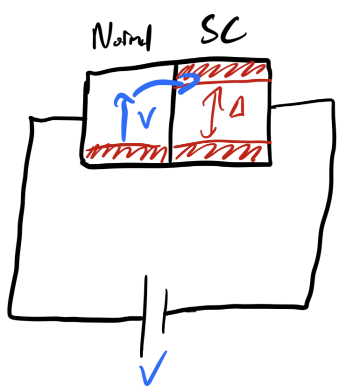
What do the excitations look like? If we go back to the boson wavefunction:

$$|\psi_{\text{boson}}\rangle = e^{b_0^\dagger N_0 + \sum_{\mathbf{p}} \frac{v_p}{u_{\mathbf{p}}} b_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger} |0\rangle \tag{9.27}$$

we can see that we have $\mathbf{p}, -\mathbf{p}$ pairs kicked out of the condensate.

9.5 Back to BCS - Tunnelling

We picture the following:



Therein, we need to apply a voltage to the normal metal in order for the electrons to be able to access the excited/quasiparticle states. The derivative of the current looks like:

$$\frac{\mathrm{d}I}{\mathrm{d}V} = \frac{2\pi}{\hbar}|T|^2 \frac{\mathrm{d}}{\mathrm{d}V} \int \rho_s(\xi)\rho_N(\chi) \left[f_{\text{normal}}(\chi + eV) - f_{\text{SC}}(\xi)\right] d\xi \tag{9.28}$$

The derivative becomes:

$$\frac{\partial f_N}{\partial d\xi} = -\delta(\xi + evaluate) \tag{9.29}$$

Thus we obtain:

$$\frac{\mathrm{d}I}{\mathrm{d}V} = \frac{2\pi}{\hbar^2}|T|^2 \int \rho_s(\xi) N(E_F)(-\delta(\xi + eV)) \tag{9.30}$$

Now we need to determine the density of states appearing in the above expression:

$$\frac{\mathrm{d}N}{\mathrm{d}E} = \frac{\mathrm{d}N}{\mathrm{d}\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}E} \tag{9.31}$$

with:

$$E = \sqrt{(\epsilon - \mu)^2 + \Delta^2} \tag{9.32}$$

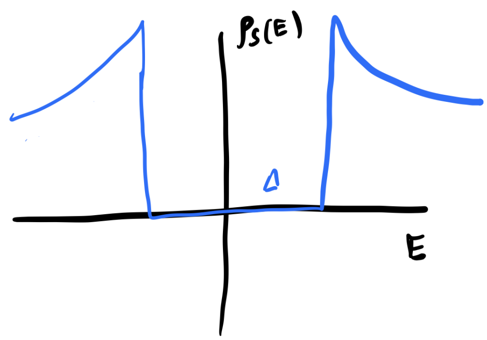
then:

$$\frac{\mathrm{d}E}{\mathrm{d}\epsilon} = \frac{\epsilon - \mu}{E} = \frac{\sqrt{E^2 - \Delta^2}}{E} \tag{9.33}$$

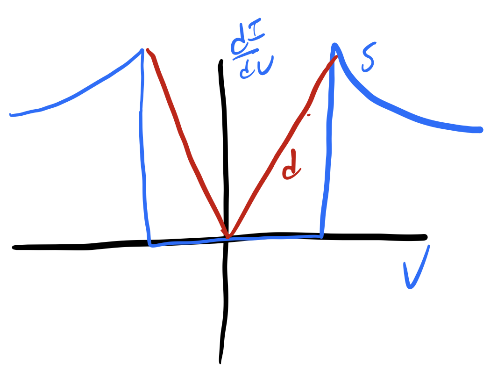
and so:

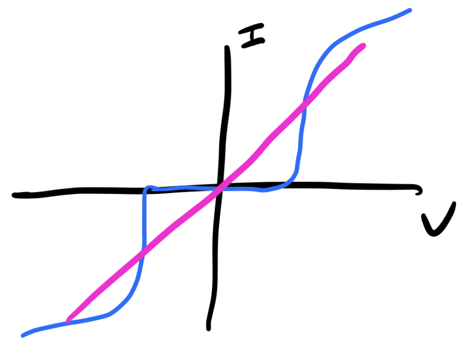
$$\rho_S(E) = N(E_F)\frac{E}{\sqrt{E^2 - \Delta^2}} \tag{9.34}$$

(for $E > \Delta$). We thus have the following DOS:



Which carries over to the shape of the $\frac{\mathrm{d}I}{\mathrm{d}V}$ vs. V curve. This is for the s-wave superconductor, for a d-wave we don't have such a steep jump (instead $\frac{\mathrm{d}I}{\mathrm{d}V}$ looks more linear):



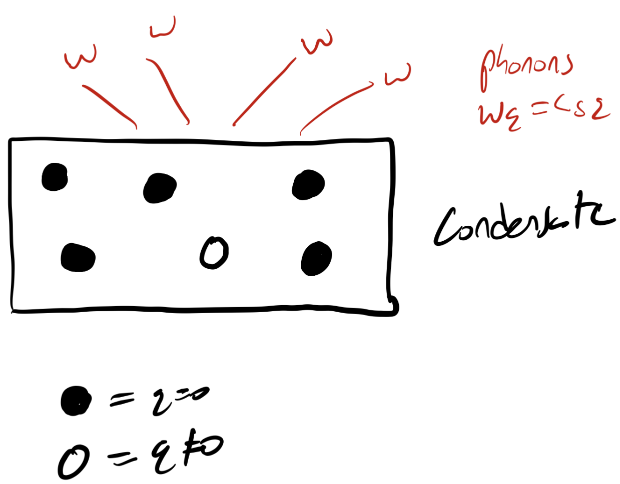


10 Bose Gases and Perturbative Calculations in BCS

Today, we will give a physical picture of the Bose gas story, revisiting depletion in Bose condensation. We will then do a perturbative calculation in BCS theory.

10.1 Depletion in Bose Theory

We have the following picture of the Bose condensate:



$$\begin{aligned} N - N_0 = \text{depletion} &= \left\langle \sum_q b_q^\dagger b_q \right\rangle = \left\langle \sum_q (u_q \alpha_q^\dagger - v_q \alpha_{-q})(u_q \alpha_q - v_q \alpha_{-q}^\dagger) \right\rangle \\ &= \sum_q \left(u_q^2 \left\langle \alpha_q^\dagger \alpha_q \right\rangle + v_q^2 [1 + \left\langle \alpha_{-q}^\dagger \alpha_{-q} \right\rangle] \right) \end{aligned} \tag{10.1}$$

The expectation values are only nonzero when $T \neq 0$. At $t = 0$, we only have the quantum depletion

$$(N - N_0)_{T=0} = \sum_q v_q^2 = \mathcal{O}(V(0)^{3/2}) \tag{10.2}$$

with $V(0)$ the repulsion. So, it must be small in order to have self-consistency. The other terms correspond to thermal depletion, coming from (at $T \neq 0$):

$$\left\langle \alpha_q^\dagger \alpha_q \right\rangle = \frac{1}{e^{\beta E_q} - 1} \tag{10.3}$$

The phonons are sound modes with $\omega_q = c_s q$. These are soft excitations. In fact they are Goldstone bosons, here collective quasi particles that interact with the condensate. There is a theorem that requires that the collective modes are soft/linear.

There is one less end in our bose condensate; we dropped $V(0)$:

$$V(0) b_0^\dagger b_0^\dagger b_0 b_0 \to V(0) N_0^2 \tag{10.4}$$

we also dropped the Hartree term:

$$\sum_k V(0) b_0^\dagger b_0^\dagger b_k b_{-k} \tag{10.5}$$

The depletion is:

$$N - N_0 = \sum_{p \neq 0} b_p^\dagger b_p \tag{10.6}$$

so:

$$N_0 = N^2 - 2N \sum_p b_p^\dagger b_p + (\text{dropped}) \tag{10.7}$$

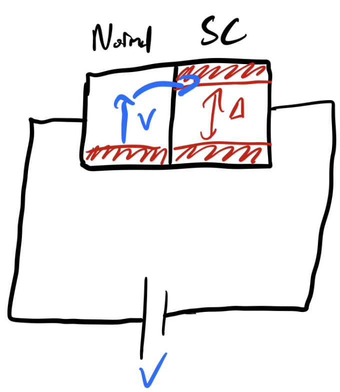
So the term we dropped is:

$$V^{\text{mf}}(0) \equiv \frac{1}{2} V(0) N_0^2 + \frac{N_0}{2} \sum V(0) (b_q^\dagger b_q + b_{-q}^\dagger b_{-q}) = \frac{1}{2} V(0) N^2 \tag{10.8}$$

which is an unimportant constant.

10.2 Tunneling

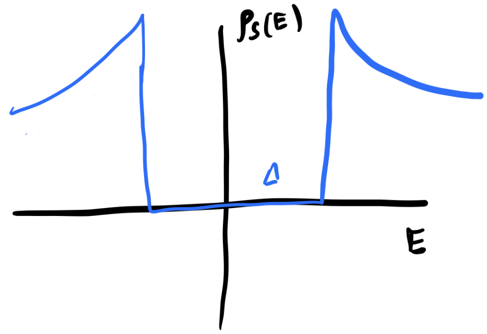
We considered the N-S tunneling setup:



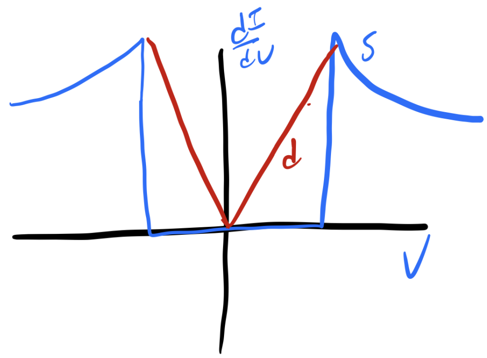
We found that:

$$\frac{\mathrm{d}I}{\mathrm{d}V} \sim \rho_s(E)\big|_{E=eV} \tag{10.9}$$

where the density of states looks like:



and the IV-curve looks like:



The crcial number is:

$$\rho_s(E) \equiv N(E) = \frac{E}{\sqrt{E^2 - \Delta^2}} \tag{10.10}$$

for $E > \Delta$.

We consider an external field:

$$H^{\text{ext}} = \Phi^{\text{ext}} c_k^\dagger c_k \tag{10.11}$$

We use Fermi's golden rule $\frac{\mathrm{d}N_{kk'}}{\mathrm{d}t}$ to compute the transition probability. But there is an assumption that needs to be made, namely that u_k, v_k don't change with the perturbation (note that this response is not gauge invariant, it depends on the choice of the gauge - hence the error in the BCS paper).

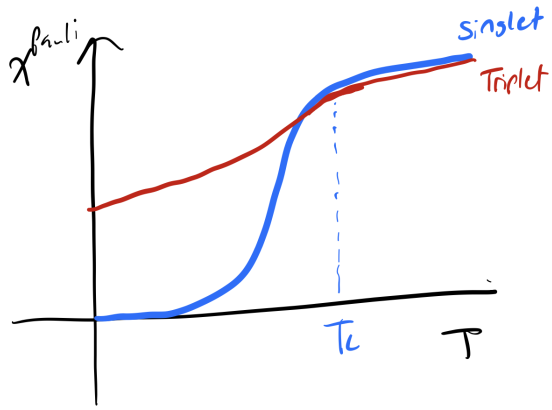
We will look at the nuclear relaxation $\frac{1}{T_1}$, the ultrasound absorption, and the real conductivity $\text{Re}(\sigma) \propto \frac{\mathrm{d}E}{\mathrm{d}t}$ (proportional to energy absorption).

This is a 1-body perturbation, so there's no reason we couldn't redo BCS theory, changing u, vs w.r.t the Φ^{ext}. What do all of these teach us? It tells us about the fermionic changes at different temperatures.

Before we do all of these, we look at the easiest thing - the Pauli susceptibility.

10.3 Pauli Susceptibility of Superconductor

The Pauli susceptibility we expect to start to fall at T_c and go to zero at $T = 0$ (at least for the singlet state):



We look at:

$$\chi = \lim_{H\to 0} \frac{\partial M}{\partial H} \tag{10.12}$$

with:

$$M = \sum_k N_{k\uparrow} - \sum_k N_{K\downarrow} = \sum_k \left\langle c_k^\dagger c_{k\uparrow} \right\rangle - \sum_k \left\langle c_{-k\downarrow}^\dagger c_{-k\downarrow} \right\rangle \tag{10.13}$$

It turns out that:

$$u_k(H) = u_k(0) + O(H^2) \tag{10.14}$$

$$v_k(H) = v_k(0) + O(H^2) \tag{10.15}$$

i.e. for linear response the coefficients do not change to first order in the external field. The energy does shift:

$$E_{h\sigma} = E_k^0(0) - \mu_B \sigma H \tag{10.16}$$

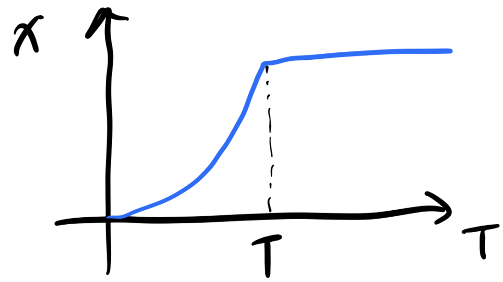
So, let's compute things:

$$\begin{aligned} \langle M \rangle &= \sum_k \left\langle (u_k \alpha_{k\uparrow}^\dagger - v_k \alpha_{-k\downarrow})(u_k \alpha_{k\uparrow} + v_k \alpha_{-k\downarrow}^\dagger) \right\rangle - \sum_k \left\langle (u_k \alpha_{k\uparrow}^\dagger - v_k \alpha_{k\uparrow})(u_k \alpha_{-k\downarrow} + v_k \alpha_{k\uparrow}^\dagger) \right\rangle \\ &= \sum_k (u_k^2 f_{k\uparrow} + v_k^2 (1 - f_{k\downarrow})) - \sum_k (u_k^2 f_{k_\downarrow} + u_k (1 - f_{k\uparrow})) \\ &= \sum_k f_{k\uparrow} - f_{k\downarrow} \\ &= \sum_k \frac{1}{e^{\beta E_{k\uparrow}} + 1} - \frac{1}{e^{\beta E_{k\downarrow}} + 1} \end{aligned} \tag{10.17}$$

Now with $E_{k\sigma} = E_k - \sigma \mu_B H$ with $E_k = \sqrt{\xi^2 + \Delta^2}$ we have:

$$\frac{\partial M}{\partial H} = \sum_k \frac{\partial f}{\partial E} \frac{\partial E_{k\uparrow}}{\partial H} - \sum_k \frac{\partial f}{\partial E} \frac{\partial E_{k\downarrow}}{\partial H} = 2 \sum_k \frac{\partial f}{\partial E} (-\mu_B) \tag{10.18}$$

So we have:



10.4 BCS and Fermi's Golden Rule

So that's the simplest calculation done. Note that as we go along, we will see the divergence in the DOS crop up:

$$\frac{\partial N}{\partial E} \sim \frac{E}{\sqrt{E^2 - \Delta^2}} \tag{10.19}$$

Note that when we look at $\omega \neq 0$ experiments, we will see both pair creation $\alpha^\dagger \alpha^\dagger$ and pair scattering $\alpha_k^\dagger \alpha_k$. The pair creation will be particularly important when we look at the AC conductivity.

We will also want to look at the coherence factors; they will end up cancelling some of the singularities. $\alpha\alpha'$ will appear as $uu' \pm vv'$. $\alpha\alpha^\dagger$ will arise as $uv' \pm vu'$.

So. some experiments will probe the singularity, in others the singularities get cancelled.

A general perturbation looks like:

$$B(k'\sigma', k\sigma)(c^\dagger_{k'\sigma'} c_{k\sigma} \pm c^\dagger_{-k-\sigma} c_{-k'-\sigma'}) \tag{10.20}$$

Note that:

$$\frac{\mathrm{d}E}{\mathrm{d}t} \propto \mathrm{Re}\sigma \leftrightarrow \omega(\frac{\mathrm{d}N_{kk'}}{\mathrm{d}t} - \frac{\mathrm{d}N_{k'k}}{\mathrm{d}t}) \tag{10.21}$$

So looking for the conductivity:

$$\begin{aligned} \mathrm{Re}\sigma \propto \frac{1}{2} N(E_F) \int d\xi \int d\xi' |B|^2 \Big((\text{coherence factor})(f - f')[\delta(\omega + E - E') + \delta(\omega + E' - E)] \Big) \\ + \frac{1}{2} N(E_F) \int d\xi \int d\xi' |B|^2 \Big((\text{coherence factor})(1 - f - f')[\delta(\omega - E - E') + \delta(\omega + E' + E)] \Big) \end{aligned} \tag{10.22}$$

Now, let us convert the integrals:

$$\int d\xi \int d\xi' \to \int dE \int dE' \frac{E}{\sqrt{E^2 - \Delta^2}} \frac{E'}{\sqrt{E'^2 - \Delta^2}} \delta(E - E') = \int dE \frac{E^2}{E^2 - \Delta^2} \tag{10.23}$$

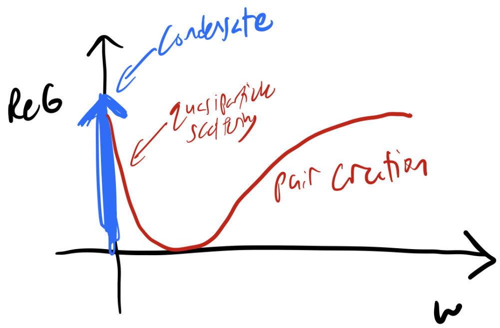
Which has the divergence at the gap, but we will see that the coherence factors carry the exact same divergences. For the quasiparticle scattering:

$$\frac{1}{2}(uu' + vv')^2 = \frac{EE' + \Delta^2}{2EE'} \tag{10.24}$$

The pair creation gives:

$$\frac{1}{2}(uv' - vu')^2 = \frac{EE' - \Delta^2}{2EE'} \tag{10.25}$$

So the pair creation gets rid of the singularity, and the quasiparticle scattering remains singular. We can see this in the conductivity plot:



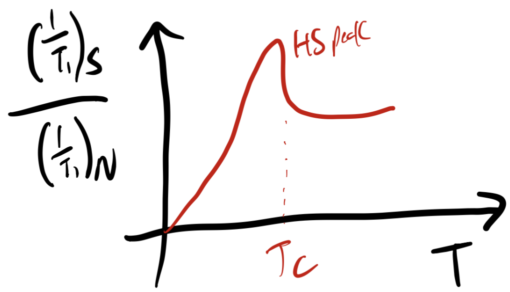
When we go to NMR and ultrasound, we will see the same features.

10.5 NMR + Ultrasound

If we look at NMR relaxation, $\frac{1}{T_1}$ corresponds to the spin flip process of conduction electrons. We have the effective field:

$$H^{\text{eff}} = gI \sum_{kk'} c^\dagger_{k'\uparrow} c_{k\downarrow} \tag{10.26}$$

There is no cancellations of divergences. If we plot the ratio of T_1s between the normal and superconducting states, we see the Hebel-Schlicter peak:

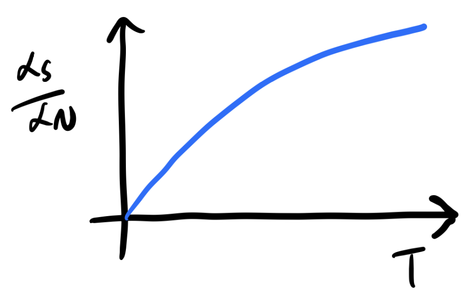


The message is that we could see how the conductivity changed as the conduction electrons were turned on - this is an excitation effect rather than a probe of the condensate.

In ultrasound, we look at the external field:

$$H^{\text{ext}} = g \sum_{qk\sigma} (a_q + a^\dagger_{-q}) c^\dagger_{k\sigma} c_{k'\sigma} \tag{10.27}$$

and if we look at the ratio of ultrasound attenuations:



we see that the divergences indeed get cancelled.

Later, we will look at more interesting consequences in a more consistent way. We will look at time-dependent response formula (like Kubo) so how we can see how BCS admits a Meissner effect, as well as Josephson tunnelling. Another way to get to the physics is to get into Landau-Ginzberg theory.

11 Landau-Ginsberg and Gross-Pitaevskii theory

Last lecture, we looked at BCS fermions in the presence of external perturbations, looking at:

1. I vs. V, which mirrored the density of the excitations $N(E)$.
2. χ^{Pauli} vs T
3. $\text{Re}(\sigma(\omega))$ vs. ω. We saw different parts of the conductivity plot corresponding to quasiparticle scattering and pair creation.
4. NMR (we saw a HS peak due to $N(E) \sim \frac{E}{\sqrt{E^2-\Delta^2}}$) and ultrasound (no singularity) vs. T

Later on we will return to fermions (using BCS), looking at Josephson tunnelling and the Meissner effect. These will be slightly more difficult to understand, so as an intermediate: today we look at the condensate, in both bosonic and fermionic contexts. We will look at Landau-Ginzberg theory for fermions and Gross-Pitaevskii theory for bosons. We will study the nonlinear Schrodinger equations that emerge as a result.

In the $\sigma(\omega)$ spectrum, the condensate lives at $\omega = 0$ as a delta function peak.

11.1 Landau-Ginsberg Theory

We look at:

$$\Omega^S - \Omega^N = A|\Delta|^2 + \frac{B}{2}|\Delta|^4 + C|\mathbf{\nabla}\Delta|^2 \tag{11.1}$$

with the physical solution:

$$\frac{\partial\Omega}{\partial\Delta} = 0. \tag{11.2}$$

where:

$$A = N(E_F)\left(\frac{T-T_c}{T_c}\right) \tag{11.3}$$

and:

$$B = \frac{0.1066 N(E_F)}{(k_B T_c)^2} \tag{11.4}$$

and:

$$C = N(E_F)\xi_0^2 \tag{11.5}$$

with:

$$\xi_0 = \frac{\hbar v_F}{\pi\Delta(T=0)} \sim 1000\text{Angstroms} \tag{11.6}$$

11.2 Heat Capacity, Correlation functions using LG theory

We can do simple calculations (via magnetism) to find ΔC_V at T_c, and we would find:

$$\Delta C_v = -T\,\frac{\partial^2 \Omega_{\text{phys}}}{\partial T^2} \tag{11.7}$$

The physical solution (dropping the gradient term) is:

$$2|\Delta|A + 2B|\Delta|^3 = 0 \implies \Delta^2 = -\frac{A}{B}, \quad \Delta\Omega = A\left(-\frac{A}{B}\right) + \frac{B}{2}\left(\frac{A}{B}\right) \implies \Delta\Omega_{\text{phys}} = -\frac{A^2}{2B} \propto (T - T_c)^2 \tag{11.8}$$

So then:

$$\Delta C_V \to -T\,\frac{\partial^2}{\partial T^2}\,(T - T_c)^2 \tag{11.9}$$

so then:

$$\left.\frac{(\Delta C_V)_S}{(\Delta C_V)_N}\right|_{T_c} = 1.43 \tag{11.10}$$

We can also calculate:

$$\langle \Delta(r)\Delta(0) \rangle = \frac{e^{-r/\xi(T)}}{r} \tag{11.11}$$

with:

$$\xi^2(T) = \frac{\xi_0^2}{\frac{T}{T_c} - 1} \tag{11.12}$$

11.3 Adding a magnetic field

The above were what we get without much work. But for things like susceptibility, we need to add a magnetic field, wherein the gradient term becomes:

$$|\nabla \Delta|^2 \to |(\nabla - \frac{2e\mathbf{A}}{c})\Delta|^2 \tag{11.13}$$

where $\mathbf{A}$ must be self-consistent (we apply an external field, but there is an internal field inside of the system which it can adjust/must be self-consistent. This is what we solve for).

So, let's include a magnetic field and see what happens. We have, largely, 3 effects in LG:

1. Condensation energy gain (in the A, B terms)
2. Kinetic energy (from the C/gradient term)
3. Field expulsion (we put in this term by hand). This is a field free-energy effect of $\int \frac{H^2(r)}{8\pi} d^3r$.

In a bose gas this is much easier because we don't have a magnetic field analog - though, we only know how to work at $T = 0$ there. Here, instead we are restricted to be around $T = T_c$.

The difference in the thermodynamic potential looks like, with these three terms:

$$\Omega_S - \Omega_N = \int A|\Delta|^2 + B|\Delta|^4 + C|(-i\nabla - \frac{2e\mathbf{A}}{c})\Delta|^2 + \frac{H^2}{8\pi} d^3r \tag{11.14}$$

Now we define something that looks like a wavefunction:

$$\psi(r) = \frac{\sqrt{2m^*c}}{\hbar}\Delta(r) \tag{11.15}$$

$$\alpha = \frac{\hbar^2}{2m}\frac{\Delta}{C} \tag{11.16}$$

and:

$$\beta \equiv \left(\frac{\hbar}{2m^*}\right)^2 \frac{B}{c^2} \tag{11.17}$$

m^* is a made up/artificial quantity to make the above equation look like a Schrodinger equation. With this we have:

$$\Omega_S - \Omega_N = \int d^3r\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}|(-i\boldsymbol{\nabla} - \frac{2e\underset{\sim}{A}}{c})\psi|^2 + \frac{(\boldsymbol{\nabla}\times\mathbf{A})^2}{8\pi} \tag{11.18}$$

We have two variational parameters $\psi(r), \mathbf{A}(r)$, the former which we can call the "pair wavefunction". To determine these, we consider a variational procedure wherein $\psi^* \to \psi^* + \delta\psi^*$ (we can vary either ψ or ψ^*) and $\mathbf{A} \to \mathbf{A} + \delta\mathbf{A}$, then regroup terms such that we get:

$$\delta(\Omega_S - \Omega_N) = 0 = (\ldots)\delta\psi^* + (\ldots)\delta\mathbf{A} \tag{11.19}$$

and we thus get two LG equations.

With $\mathbf{A} = 0$, we have:

$$\delta\Omega = \int \left[\alpha\psi + \beta|\psi|^2\psi - \frac{\boldsymbol{\nabla}^2\psi}{2m^*}\right]\delta\psi^* + \tag{11.20}$$

where we obtained the $\boldsymbol{\nabla}^2$ via integration by parts.

This should vanish for any variation, and hence (adding back the magnetic field term):

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}(-i\boldsymbol{\nabla} - \frac{2e\mathbf{A}}{c})^2\psi = 0 \tag{11.21}$$

which looks like a nonlinear schrodinger equation.

For the $\mathbf{A}$ equation, the field potential term gives:

$$\int(\boldsymbol{\nabla}\times\mathbf{A})(\boldsymbol{\nabla}\times\delta\mathbf{A}) \to \int *(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))\delta A \tag{11.22}$$

so in combination with the gradient term we obtain:

$$J_s = \frac{e}{im^*}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) - \frac{4e^2}{m^*c}|\psi|^2\mathbf{A} \tag{11.23}$$

which is an equation for the supercurrent. The first term is the paramagnetic term, the second term is the diamagnetic term.

We can't interpret this wavefunction as $P(x) = |\psi(x)|^2$, but in some sense we can interpret it as a kind of classical field (with quantum origins) which does tell us about the properties/dynamics. We may also be bothered by the fact that we have a m^* appearing the equations, which has no physical implications....

Let us write $\psi = |\psi|e^{i\theta}$. Then we can write:

$$J_s^{\text{para}} = \frac{2e}{m^*}|\psi|(\boldsymbol{\nabla}\theta) \tag{11.24}$$

so the supercurrent is being driven by a gradient of phase. We can write the diamagnetic part as:

$$J_s^{\text{dia}} = \frac{(2e)^2|\psi|^2\mathbf{A}}{m^*c} \tag{11.25}$$

11.4 Gross-Pitaevskii for Bosons

We turn to neutral superfluids bose condensates. How do things change there? Note that in this setting, the theory only works at $T = 0$, and we have a real mass (c.f. Fermi condensate theory works at $T \approx T_c$ with artifical m^*). We derive the Gross-Pitaevskii equations in this context.

We take solutions of the form:

$$\psi(r) = \sqrt{N_0}\phi(r) \tag{11.26}$$

and define:

$$|n(r)| \equiv |\psi(r)|^2 \tag{11.27}$$

We then have the energy functional:

$$E[\phi(r)] = N_0 \int d^3r \left(\frac{\hbar^2}{2m}\right) |\nabla\phi|^2 + \frac{1}{2}N_0^2 U_0 |\phi(r)|^4 - \mu N_0 |\phi(r)|^2 \tag{11.28}$$

where the second term comes from the 2-body interaction and $U_0 = V(0)$. We now have to self-consistently solve this from the variational conditions:

$$\phi^* \to \phi^* + \delta\phi^* \tag{11.29}$$

and then we find the equation of motion:

$$\frac{\delta E}{\delta \phi^*} = 0 \tag{11.30}$$

This yields the Gross-Pitaevskii equation:

$$\left[-\frac{\nabla^2}{2m} + U_0|\psi(r)|^2\right]\psi(r) = \mu\psi(r) \tag{11.31}$$

which is a nonlinear Schrodinger equation, with the $U_0|\psi(r)|^2\psi(r)$ being the analog to the β term in the fermionic case.

I can certainly write down a current (though it does not come out of a variational condition as in the fermion case), just derived from the GP equation:

$$J_s = \frac{i}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) \tag{11.32}$$

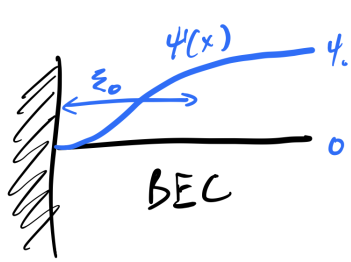
which is the particle flow current, and with $\psi = N_0 e^{i\theta}$:

$$J_s = \frac{N_0 \nabla\theta}{m} \tag{11.33}$$

which implies $\nabla \times v_s = 0$, i.e. the fluid is irrotational, associated with zero viscosity.

11.5 Gross-Pitaevskii in 1-D

We consider a BEC in 1-D, with a hard wall on the left at $x = 0$. We then have the boundary conditions of $\psi(x = 0)$ and $\psi(x \to \infty) = \psi_0$.



We have the equation of motion:

$$\frac{\hbar}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + U_0|\psi|^2\psi = \mu\psi \tag{11.34}$$

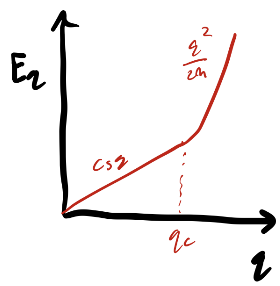
with $|\psi|^2 = N_0$, so then we have a length scale:

$$\xi_0^2 = \frac{\hbar}{2mU_0N_0} = \frac{1}{q_c^2} \tag{11.35}$$

and solution:

$$\psi = \psi_0 \tanh\frac{x}{\xi_0\sqrt{2}} \tag{11.36}$$

This length scale appears in Bogoliobov theory, where there is a crossover from linear/acoustic to quadratic/free behaviour:



We can use GP to derive Bogoliobov, but we need to get the time-dependent equation first.

11.6 Time-dependent GP equation

We write:

$$H - \mu N = \int \frac{1}{2m}|\nabla\psi|^2 + \frac{U_0}{2}\int \psi^*\psi^*\psi\psi - \mu\psi^*\psi \tag{11.37}$$

and then writing down the Heisenberg equation of motion:

$$i\frac{\partial\psi}{\partial t} = -[H, \psi(r)] \tag{11.38}$$

we then find:

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2m}\nabla^2\psi + U_0\psi^*\psi\psi - \mu\psi \tag{11.39}$$

which is the time-dependent GP equation. We solve these time-dependent equations asserting that:

$$\delta\psi = e^{-i\mu^*}(ue^{-i\omega t} - v^*e^{i\omega t}) \tag{11.40}$$

We will then find that ω is the Bogoliobov condensate. We solve this via two equations of motion:

$$\begin{aligned} i\frac{\partial\delta\psi}{\partial t} &= \frac{1}{2m}\nabla^2(\delta\psi) + U_0(2|\psi_0|^2\delta\psi + \psi_0^2\delta\psi^*) \\ -i\frac{\partial\delta\psi}{\partial t} &= \frac{1}{2m}\nabla^2(\delta\psi^*) + U_0(2|\psi_0|^2\delta\psi^* + \psi_0^2\delta\psi) \end{aligned} \tag{11.41}$$

then diagonalize via u, vs and then we find to Bogoliobov dispersion:

$$\omega = \sqrt{\epsilon^2(p) + 2\epsilon(p)N(0)U_0} \tag{11.42}$$

So the takehome message; the collective modes of the Bose gas is connected with the excitation spectrum. This is not true of the Fermi gas, where the pairs are split.

12 Landau-Ginsberg II, The Proximity Effect

12.1 Review + Overview

For Fermi condensates, we had the Landau-Ginsberg equation:

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}(-i\nabla - \frac{2e\mathbf{A}}{c})^2\psi = 0 \tag{12.1}$$

coupled with the current:

$$J_s = \frac{e}{im^*}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{4e^2}{m^*c}|\psi|^2\mathbf{A} \tag{12.2}$$

C.f. the Gross-Pitaevski equation for describing Bose condensates:

$$\left[-\frac{\nabla^2}{2m} + U_0|\psi(r)|^2\right]\psi(r) - \mu\psi(r) = 0 \tag{12.3}$$

Characteristically, we have a length scale (for the Fermi case):

$$\xi_0 = \frac{\hbar v_F}{\pi\Delta(0)} \tag{12.4}$$

And the temperature-dependent coherence length is:

$$\xi^2(T) = \frac{1}{2m|\alpha|} \tag{12.5}$$

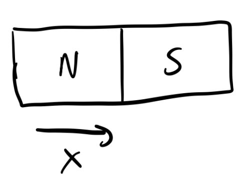
which we will show.

The examples we will study in the next two lectures are:

1. The proximity effect (no fields)
2. Meissner effect in type-II superconductors
3. Critical current in very thin films
4. Flux quantization
5. H_{CZ} upper critical field and vortices

12.2 The Proximity Effect

In the proximity effect, we take a normal metal and place it next to a superconductor. Normal could be a ferromagnet, or a topological insulator etc. We then ask when $\mathbf{A} = 0$ what is $\psi(x)$ through the sample?



We solve:

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m}(-\nabla^2\psi) = 0 \tag{12.6}$$

so ψ penetrates the normal metal over length scale ξ. The different regions are identified via boundary conditions on the LG equations. This is the same equation we did before for the Bose gas. We write $\psi = \psi_0 f(x)$ with ψ_0 the homogenous solution, so $\psi_0^2 = \frac{-\alpha}{\beta}$.

Then:

$$-\frac{1}{2m\alpha}\frac{\partial^2 f}{\partial x^2} + f - f^3 = 0 \tag{12.7}$$

This implies:

$$f = \tanh(\frac{x}{\sqrt{2}}\frac{1}{\xi(T)}) \tag{12.8}$$

With:

$$\alpha = \frac{1}{2m}\frac{A}{C} \tag{12.9}$$

$$A = N(E_F)\left(\frac{T - T_c}{T_c}\right) \tag{12.10}$$

$$C = N(E_F)\xi_0^2 \tag{12.11}$$

$$\xi^2(T) = \frac{1}{2m|\alpha|} \tag{12.12}$$

Thus we can write Eq. (12.7) as:

$$\frac{1}{2m|\alpha|}\frac{\partial^2 f}{\partial x^2} + f - f^3 = 0 \tag{12.13}$$

We can thus see the leaking of superconductivity.

How do we visualize this? It's easier to see in Bogoliubov de Gennes.

12.3 Bogoliubov de Gennes Theory

We want to think about the spatial dependencies of $\Delta(r), V(r), \mathbf{A}(r), \ldots$. Is there a way to think about the coherence parameters $v(r), u(r)$ in such a way that holds on to ideas from BCS theory?

There indeed is; we go back to the BCS diagonalization:

$$\begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix}\begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} \tag{12.14}$$

Let us now try to solve the same problem with a nonlocal interaction/spatial dependence:

$$H_0(r) = \nabla^2(\ldots) + \mathbf{A}(r)\cdot\mathbf{J} + V(r)\rho \tag{12.15}$$

We then get the equations:

$$\begin{aligned} H_0(r)u_n(r) + \Delta(r)v_n(r) &= E_n u_n(r) \\ -H_0^*(r)v_n(r) + \Delta^*(r)u_n(r) &= E_n v_n(r) \end{aligned} \tag{12.16}$$

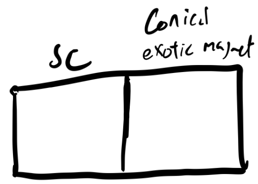
Which really can only solved numerically. Note that we are *not* constructing eigenstates $\prod(u_n + v_n c^\dagger c^\dagger)|0\rangle$. The key equation is instead the gap:

$$\Delta(r) = V\sum_n v_n^* u_n \tanh(\beta\frac{E_n}{2}) \tag{12.17}$$

This says nothing about what we pair. What the output/final result of this feature is the superconducting gap and how it may vary throughout space (i.e. we can tell how the superconductor persists throughout space) (we have an input gap, but we get out how the gap really varies through space).

12.4 Superconducting spintronics and Majoranas

Another example of a proximity-effect-like effect; we put a conical exotic magnet with charcteristic $\mathbf{Q}$ vector next to a superconductor.

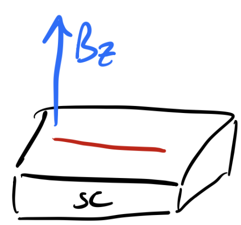


Then we induce a superconducting gap:

$$\Delta_{\mathbf{Q}} \sim \left\langle c_{\mathbf{k}}^{\dagger} c_{\mathbf{Q}-\mathbf{k}}^{\dagger} \right\rangle \tag{12.18}$$

which allows us to get triplet spin pairs and currents. This is a very exotic kind of superconductivity; LOFF (named after the inventors). This is interesting in the sense that the Meissner effect does not shut off the current. There are two competing lengths here, $\xi(T)$ vs. $\lambda(T)$ (the penetration).

There is a recipe where if you have a 1D superconductor with $p + ip$ spinless pairs, it will host Majorana fermions. 2 ways to do it; 2D superconductor with a 1D wire on top, made of a semiconductor with spin-orbit coupling, and apply an external field B_z. Then by proximity we could get Majoranas on the edge.

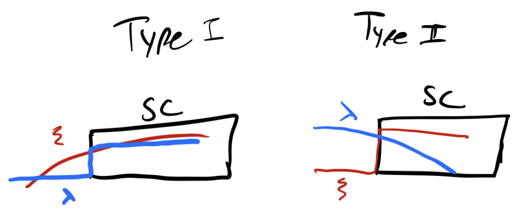


The other way is to have a thin wire on top of a 2-D electron gas.semiconductor. Then we get a channel between two Josephson phases. There has been a huge experimental effort in trying to realize/measure majorana zero modes. In 2018 it was supposedly found - it was an indirect probe, but the idea is that there should be quantized conductance; the paper was retracted a year later. But there is an idea that there is this recipe for constructing majoranas, all based on the proximity effect.

12.5 The Meissner Effect (via LG)

Recall there are two types of superconductivity, type-I and type-II. There are two characteristic lengths - the penetration depth $\lambda(T)$ as well as the coherence length $\xi(T)$. In type I we have $\lambda \ll \xi$ and in type II we have $\xi \ll \lambda$.

If we draw a picture:



Today, we look at Meissner in type-II. Here we have that $\psi \approx$ const. inside the superconductor, wherein the grad terms drop out of the current and so:

$$\mathbf{J}_s \approx -\frac{4e^2}{mc}|\psi_0|^2\mathbf{A} \tag{12.19}$$

Now if we take the curl of both sides:

$$\boldsymbol{\nabla} \times \mathbf{J}_s = -\frac{4e^2}{mc}|\psi_0|^2\mathbf{H} \tag{12.20}$$

Then using that $\mathbf{J}_s = \frac{4\pi}{c}\boldsymbol{\nabla} \times \mathbf{H}$ we find:

$$-\frac{4\pi}{c}\boldsymbol{\nabla}^2\mathbf{H} = -\frac{4e^2}{mc}|\psi_0|\mathbf{H} \tag{12.21}$$

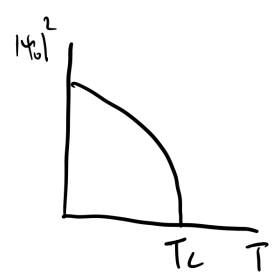
So then if we choose our axis system such that: $\mathbf{H} = H\hat{\mathbf{z}}$:

$$\frac{\partial^2 H}{\partial z^2} = \left(\frac{1}{\lambda_L}\right)^2 H \implies H = H_0 e^{-\frac{z}{\lambda_L}} \tag{12.22}$$

With:

$$\left(\frac{1}{\lambda_L}\right)^2 = \frac{4\pi}{c}\left[\frac{4e^2}{mc}|\psi_0|^2\right] \tag{12.23}$$

where $|\psi_0|^2$ is a temperature dependent superconducting order parameter/superfluid density:

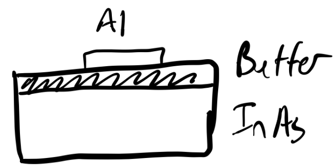


It turns out that the story is a little more complicated - it is energetically beneficial to create vortices, and not completely expel the magnetic field in some cases.

13 Landau-Ginsburg III

13.1 New progress on Topological SC/Majoranas?

Last Wednesday - Microsoft Majoranas? Maybe worth a look. We'll briefly discuss the two papers:
In Phys Rev B 107 245423 paper, they studied a system:



They looked at more complete tunneling conductors (quantized). Seems like better progress on creating materials. They looked at the tunneling matrix $T_{ij} = \dfrac{\partial I_i}{\partial V_j}$, and avoided some of the older problems.

In Nature 638, Feb 20, 2025, they discovered interferometeric evidence for Majorana fermions, and 3 "qubits".

13.2 Review + Overview

We have been studying applications of LG theory; we discussed the Meissner and proximity effect, with length scales λ_L and ξ. Today, we do more applications to finish this story. Today, we do critical current J_c in thin films, flux quantization, and upper critical field and vortices.

Recall that type I superconductors are where $\xi \gg \lambda_L$ and type II are $\xi \ll \lambda_L$. The latter is more useful, and this is what we will make a case for today.

On Friday, we will then study the Messiner effect from the perspective of BCS theory. We will use linear response/time-dependent PT, and study the Kubo formula. We will also study the Josephson effect.

13.3 Critical Current

We have the LG equations:

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m}\left(-i\nabla - 2e\mathbf{A}\right)^2\psi = 0 \tag{13.1}$$

$$\mathbf{J}_s = \frac{e}{im}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{4e^2}{mc}|\psi|^2\mathbf{A} = \frac{2e}{m}|\psi|^2\mathbf{v}_s \tag{13.2}$$

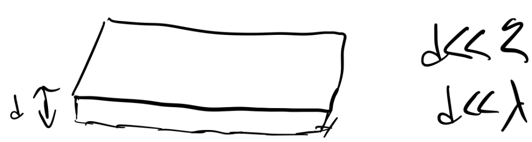
with $\psi = |\psi|e^{i\phi}$, and:

$$\mathbf{v}_s = \nabla\phi - \frac{2e\mathbf{A}}{c} \tag{13.3}$$

we have a key parameter:

$$|\alpha| = \frac{1}{2m\xi^2(T)} \tag{13.4}$$

The first question is how do we think about a current in a thin film? Infinite extension in xy-plane.



We can try to set up a constant $\mathbf{J}$; what is the max value allowed?

In this scenario of the thin film, we can say that $|\psi|$ is independent of x. Further, we can ignore $\frac{\mathbf{H}^2}{8\pi}$ - no flux expulsion. From this, we find that $\mathbf{v}_s$ is constant, and so $\mathbf{J}_s$ is constant.

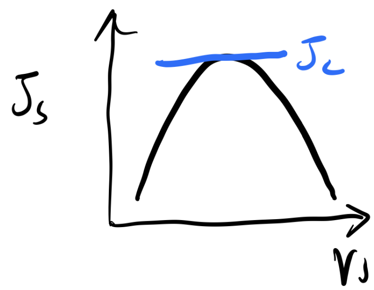
We rewrite the first equation:

$$\Omega_S = \Omega_N - |\alpha||\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{|\psi|^2}{2}m\mathbf{v}_s \tag{13.5}$$

Then with the varaitional condition $\dfrac{\partial \Omega}{\partial |\psi|} = 0$ we then get:

$$|\psi|^2 = \psi_0^2[1 - \frac{m\mathbf{v}_s^2}{2|\alpha|}] \tag{13.6}$$

so the amplitude of ψ which goes into the current is reduced. The current itself is also changes as $|\psi|$ changes:



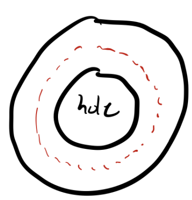
Above the critical current J_c we become normal.
Let's deduce $v_s^{\max}$ by setting $|\psi|^2$ to zero, and from this derive:

$$J_s^{\max} = \frac{2e|\psi_0|^2}{m\xi(T)} \tag{13.7}$$

which tells us that short coherence lengths $\xi(T)$ are good, because we get huge critical currents.

13.4 Flux Quantization

This is a signature of $e^* = 2e$ pairing. Consider a multiply connected superconductor:



If I expose the SC to a magnetic field, we will find that there will be trapped quantized flux. Consider a loop integral, with the loop taken far from the edges, compared to λ_L, ξ; then we find that the integral vanishes.

$$0 = \oint \mathbf{J} \cdot d\mathbf{l} \tag{13.8}$$

We then recall that $\mathbf{J} = \frac{2e}{m}|\psi|^2\mathbf{v}_s$, so:

$$0 = \oint \mathbf{J} \cdot d\mathbf{l} \implies \int \boldsymbol{\nabla}\phi \cdot d\mathbf{l} = 2\pi n \tag{13.9}$$

and therefore:

$$\int \frac{2e\mathbf{A} \cdot d\mathbf{l}}{c} = \frac{2e\Phi}{c} \tag{13.10}$$

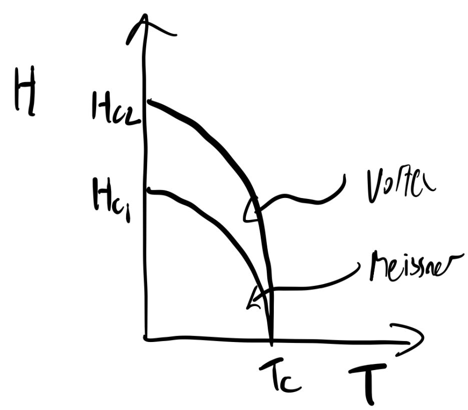
and recovering the h via dimensional analysis:

$$\Phi = \frac{hcn}{2e} \tag{13.11}$$

for $n \in \mathbb{Z}$.

13.5 Upper Critical Field

Type-II superconductors have phase diagram:



We will find $H_{C_2} \propto \frac{1}{\xi^2(T)}$, so we have a big critical field if we have a short coherence length. Let's return to the first LG equation, and calculate H_{C_2} by solving the LG equation for when ψ is small (this can be done because near the critical field the order parameter is very small). We then solve:

$$-\frac{1}{2m}(-i\nabla - \frac{2e\mathbf{A}}{c})^2\psi = |\alpha|\psi \tag{13.12}$$

with $|\alpha| = \frac{1}{2m\xi(T)^2}$. We have the energy eigenvalues:

$$E_n = (n + \frac{1}{2})\hbar\omega_c \tag{13.13}$$

with:

$$\omega_c = \frac{(2e)H_{C_2}}{mc} \tag{13.14}$$

So then taking $n = 0$ we find:

$$E_0 = |\alpha| = \frac{1}{2}\left(\frac{2eH_{C_2}}{mc}\right) \tag{13.15}$$

where then:

$$H_{C_2} = \frac{c}{2e\xi^2(T)} = \frac{\Phi_0}{2\pi\xi^2(T)} \tag{13.16}$$

Now, what $\psi(x,y)$? These contain important information about the field configurations. We have:

$$\psi(x,y) = e^{ik_y}g(x) \tag{13.17}$$

So then we solve:

$$\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{\partial}{\partial y} - \frac{2eiH_{C_2}x}{\hbar c}\right)^2 \psi + \frac{1}{\xi^2}\psi = 0 \tag{13.18}$$

I can then break this up into a term with k_y and one with $g(x)$, which we would then find:

$$g(x) = Ce^{-\frac{(x-x_0)^2}{2\xi^2}} \tag{13.19}$$

$$x_0 = \frac{k_y c}{2eH_{C_2}} \tag{13.20}$$

Abrikosov's Idea was to sum:

$$\psi(x,y) = \sum_{k_y} C_{k_y} e^{ik_y y} g(x) \tag{13.21}$$

If we assert a periodicity $k_y = nq$ in y, then we will see that a periodicity in x emerges. Hence we get a 2d (x,y) lattice in $\psi(x,y)$. But the periodicity we get out is wrong, because we need the $\beta|\psi|^2\psi$ term (and indeed (confirmed by experiment) we get a triangular lattice as as a result):

$$\psi(x,y) = \sum_n cg(x - \frac{\hbar cnq}{2eH_{C_2}})e^{inqy} = \sum_n g(x - nR)e^{iqny} \tag{13.22}$$

We want to now check if this is periodic in x. We take $x \to x + pR$:

$$\psi(x + pR, y) = \sum g(x - mR)e^{iqy(m+p)} = e^{iqpy}\sum_m g(x - mR)e^{imqy} = e^{iqpy}\psi(x,y) \tag{13.23}$$

So the wavefunction is invariant, up to a phase. Thus we conclude x/y periodicity (rectangular), but really the mistake is that we should have a triangular lattice due to the $\beta|\psi|^4$ term. We haven't showed that there are zeros yet, but it turns out that ψ vanishes at the holes/vortices. This allows H to penetrate into the vortices - the superconductor does not fully expel the fields in this regime.

Next Friday, we will think about time-dependent perturbation theory and linear response theory.

14 BCS Meissner Effect

14.1 Overview

Last lecture, we completed our discussion of Landau-Ginsberg theory, which is relevant near $T \approx T_c$. We found:

- Critical current $J_c \sim \frac{1}{\xi(T)}$
- Critical field $H_{C_2} \sim \frac{1}{\xi^2(T)}$
- Flux quantization $\Phi = \Phi_0$ which suggests $e^* = 2e$

Today we do the BCS calculation of the Meissner effect. We'll then end the class with Josephson effect and modern superconductivity/cuprates next week.

Also reccomended to look at Kosterlitz-Thouless theory, Majorana fermions, Spin glasses, spintronics etc.

(Nambu's Nobel acceptance speech)

There are two collective modes in superconductors needed to insure full gauge invariance. With $\psi = |\psi|e^{i\theta}$. Nambu found oscillations in θ, anmd then Higgs looked at amplitude oscillations in $|\psi|$, but this is quite hard to observe. There is another complication - the NGB is not a soft boson, the phase mode has a gap in a charged superconductor (cf. Goldstone bosons have $\omega_q \to 0$ as $q \to 0$).

14.2 Into to Meissner in BCS

Two sets of notes on Canvas - on Meissner and on time-dependent perturbation theory.

We think about how to calculate:

$$\left\langle J(q,\omega)\right\rangle = -K(q,\omega)A(q,\omega) \tag{14.1}$$

In LG we had:

$$\mathbf{J} = (\psi^* \nabla \psi + \ldots) - \frac{4e^2}{mc}|\psi|\mathbf{A} \tag{14.2}$$

with $\mathbf{J} = \nabla \times \mathbf{H}$. We then found:

$$-\nabla^2 \mathbf{H} \left(\frac{1}{\lambda_L}\right)^2 = \mathbf{H} \tag{14.3}$$

we now study this more microscopically, using the tools of linear response theory.

We note thatthe superfluid density goes as $\propto \frac{1}{\lambda_L^2}$ and further if we plot it as a function of temperature it decreases (to zero) as we go towards T_c.

Note that the current can be related to the conductivity:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \sim -i\omega \mathbf{A} \tag{14.4}$$

$$\mathbf{J}(\omega) = \sigma(\omega)\mathbf{E}(\omega) \tag{14.5}$$

14.3 Time-Dependent Perturbation Theory

We split the Hamiltonian into two pieces $H^0 + H^{\text{ext}}$, with $H^{\text{ext}} = \mathbf{A} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{A}$. We then write:

$$\left\langle \mathbf{J}(t)\right\rangle = \langle \psi_s(t)|\mathbf{J}|\psi_s(t)\rangle \tag{14.6}$$

The answer will be:

$$\left\langle \mathbf{J}(t)\right\rangle = \langle \mathbf{J}\rangle_0 - i\int \left\langle [J_I(t), H_I^{\text{ext}}(t')]\right\rangle dt' \tag{14.7}$$

With:

$$O_I(t) = e^{iH_0(t-t_0)} O e^{-iH_0(t-t_0)} \tag{14.8}$$

and $[J_I(t), H_I^{\text{ext}}(t')]$ the current-current correlator. To derive this, we go to the interaction representation, writing $\psi_s(t)$ in terms of $\psi_I(t)$:

$$i\frac{\partial \psi_s}{\partial t} = (H_0 + H^{\text{ext}})\psi_s \tag{14.9}$$

$$i\frac{\partial \psi_I}{\partial t} = H_I^{\text{ext}}\psi_I(t) \tag{14.10}$$

$$\psi_I(t) = e^{-iH_0(t-t_0)}\psi_s(t) \tag{14.11}$$

We now integrate the interaction picture Shrodinger equation:

$$\psi_I(t) = \psi_I(t_0) - i\int_{t_0}^{t} H_I^{\text{ext}}(t')\psi_I(t')dt' \tag{14.12}$$

We now linearize/approximate to first order:

$$\psi_I(t) \approx \psi_I(t_0) - i\int_{t_0}^{t} H_I^{\text{ext}}\psi_I(t_0)dt' \tag{14.13}$$

Now, joining the ψ_Is together, we have:

$$\psi_I(t) \approx \left[1 - i\int_{t_0}^{t} H_I^{\text{ext}}(t')dt'\right]\psi_I(t_0) \tag{14.14}$$

Now, using this we can rewrite the expectation value $\langle\psi_s(t)|\hat{H}|\psi_s(t)\rangle$:

$$\psi_s(t) = e^{-iH_0(t-t_0)}[1 - i\int_{t_0}^{t} H_I^{\text{ext}}(t')dt'] \tag{14.15}$$

So the punchline is we get (dropping $\mathcal{O}(H_I^2)$ terms):

$$\left\langle \mathbf{J}(t)\right\rangle = \langle\mathbf{J}\rangle_0 - i\int_{t_0}^{t}\left\langle [J_I(t), H_I^{\text{ext}}(t')]\right\rangle dt' \tag{14.16}$$

For the Josephson effect, we will look at two superconductors L/R put next to each other, with $H = H_0^L + H_0^R + T_{12}$, and study the current $J = \left\langle \frac{\mathrm{d}v_R}{\mathrm{d}t}\right\rangle$.

In principle, computing this one computes commutators between lots of creation/annihilation operators.

14.4 Applying TDPT to BCS

An important point - when we make this application in BCS theory, we replace $c^\dagger c \to \alpha^\dagger\alpha^\dagger$. We assume that u_k, v_k are unchanged by the perturbation. There is no reason a priori why this should be true. In other words, we assume that $H^0 = \sum_{k\sigma} E_k \alpha_{k\sigma}^\dagger \alpha_{k\sigma}$ is unchanged. This is only ok for $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, wherein $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$ and so $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$ - collective modes do not enter in the transverse gauge.

So, when we write $H^{\text{ext}} = \mathbf{J}\cdot\mathbf{A}$ (as BCS did), we won't get the same answer if we looked at $H^{\text{ext}} = \rho\Phi$.

Similar in particle physics - work in Coloumb gauge to work out what the particles look like explicitly. In a different gauge, naively computing the spectrum of quasiexcitations, you overcount because not all of the particles are gauge invariant.

In the transverse gauge, our external Hamiltonian looks like:

$$H^{\text{ext}} = -\frac{e}{2mc}\int d^3r\psi^\dagger(r)(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p})\psi(r) \tag{14.17}$$

with:

$$\psi(r) = \int e^{ikr}c_k \tag{14.18}$$

Then:

$$H^{\text{ext}} = -\frac{e}{2mc}\sum_{kq\sigma}\mathbf{A}(q,t)\cdot(2\mathbf{k}+\mathbf{q})c_{(k+q)\sigma}^\dagger c_{k\sigma} \tag{14.19}$$

Let's write down the current (paramagnetic and diamagnetic parts) explicitly:

$$H = J^{\text{para}} + J^{\text{dia}} \tag{14.20}$$

with:

$$J^{\text{para}} = \frac{e}{2m}\sum_{kq\sigma} c_{k\sigma}^\dagger c_{k+q\sigma}(2\mathbf{k}+\mathbf{q}) \tag{14.21}$$

$$J^{\text{dia}} = -\frac{e^2}{2m}\int \mathbf{A}(r)\psi^\dagger(r)\psi(r) \tag{14.22}$$

Let's sequence the steps without the algebra:

1. Substitute αs for cs

2. We have pair creation terms $(uv' - vu')(\alpha^\dagger \alpha'^\dagger \alpha' \alpha)$ and qp scattering terms $(uu' + vv')(\alpha^\dagger \alpha' - \alpha'^\dagger \alpha)$

3. Take commutators and average $\left\langle [\alpha\alpha^\dagger, \alpha'\alpha'^\dagger] \right\rangle$ etc. We get pair creation $(uv' - vu')^2(1 - f_k - f_{k+q})$ and quasiparticle scattering $(uu' + vv')^2(f_k - f_{k+q})$ with f_ks coming from $\left\langle \alpha_k^\dagger \alpha_k \right\rangle = f_k$.

4. Take time integrals $\int dt'$; the αs carry time dependences $\alpha_k(t') \equiv e^{iE_k t'} \alpha_k^\dagger$ etc. Then, we get:

$$\left\langle [J_I(t), J_I(t')] \right\rangle = \int_{t_0}^{t} e^{i(E_k - E_{k+q})(t-t')} dt' \tag{14.23}$$

5. We get from pair creation $\frac{1-f_k-f_{k+q}}{E_k+E_{k+q}\pm\omega}(u_k v_{k+q} - v_k u_{k+q})^2$ and we get from pair scattering $\frac{f_k-f_{k+q}}{E_{k+q}-E_k\pm\omega}(u_k u_{k+q} + v_k v_{k+q})^2$.

6. Now, for the Meissner effect I want $J = -K(0,0)A$; take $\omega \to 0$ first and then take $q \to 0$. Now, the pair creation term goes to zero, and the pair scattering part persists, with the fs becoming $\dfrac{\partial f}{\partial E}$.

7. We then get:

$$J(0,0) = \left[\frac{2}{c} \left(\frac{e}{2m} \right)^2 \sum_k \frac{4k^2}{3} \left(-\frac{\partial f}{\partial E} \right) - \frac{n}{m} \frac{e^2}{c} \right] A(0,0) \tag{14.24}$$

   where the first term is the paramagnetic current and the second term is the diamagnetic current. As $T \to 0$ the paramagnetic current goes to zero and we are just left with the paramagnetic current with $\frac{n_s}{m} = \frac{n}{m}$. As $T \to T_c$, we find that the two terms cancel, which just corresponds to there being no J/no Meissner effect in the normal state.

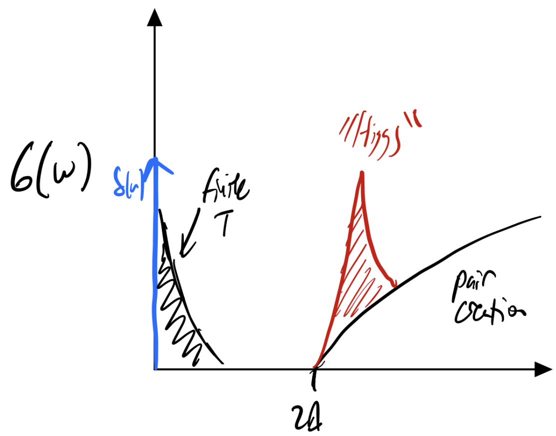
   We can write $J = -K(0,0)A$ where $K(0,0) = \frac{n_s e^2}{mc}$ where n_s is maximal at $T = 0$ and goes to zero as $T \to T_c$.

14.5 Littlewood Effect, The end of the story

How do we see the amplitude ("Higgs") mode? It happens in superconductors with $Q \neq 0$ pairs. In other words, we have $\left\langle c_{k+q}^\dagger c_{-k}^\dagger \right\rangle \neq 0$.

In NBSe, we can see $\left\langle c_{k+Q}^\dagger c_k \right\rangle$ charge density waves + superconductivity. We can see it in superconductors that carry supercurrent.

Conductivity vs. T looks like:



Monday we study the Josephson effect, but with different machinery - we will see something exotic.

15 Josephson Effect

15.1 Overview + Review

Today we discuss the Josephson effect, and on Friday we discuss exotic superconductors. Last time we discussed time-dependent linear response theory:

$$\langle \mathbf{J} \rangle = \frac{\langle \rho \rangle \mathbf{A} e^2}{mc} - i \int_{-\infty}^{t} dt' \left\langle [\mathbf{J}_I(r,t), H_I^{\text{ext}}(r,t')] \right\rangle \tag{15.1}$$

and if $H^{\text{ext}} = \mathbf{J} \cdot \mathbf{A}$ the commutator reduces to a current-current commutator at different times.

Today we look at tunnelling. Previously we looked at this via Fermi's golden rule. The setting is a normal metal next to an insulator next to a superconductor. Then:

$$I = \frac{2\pi}{\hbar} |T|^2 \int \rho_S(\xi) \rho_N * (\xi + eV) [f_N(\xi + eV) - f_S(\xi)] \tag{15.2}$$

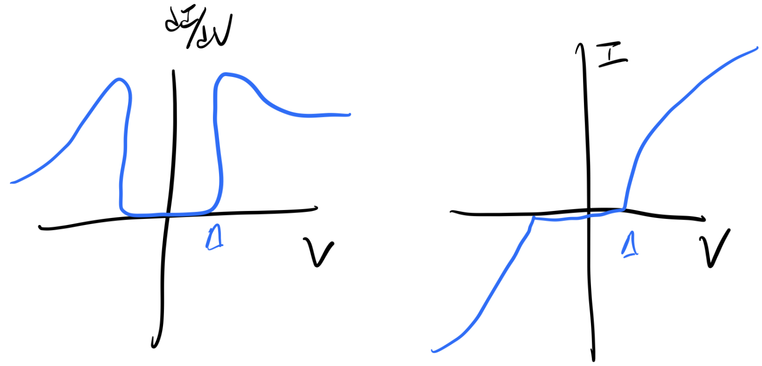
So then:

$$\frac{\mathrm{d}I}{\mathrm{d}V} = \frac{2\pi}{\hbar} |T|^2 \int \rho_S(\xi) N(E_F) \left(\frac{\partial f_N}{\partial \xi} \right) d\xi \tag{15.3}$$

taking $\frac{\partial f_N}{\partial \xi} = -\delta(\xi + eV)$:

$$\frac{\mathrm{d}I}{\mathrm{d}V} = -\frac{2\pi}{\hbar} |T|^2 \int \rho_S(\xi) N(E_F) \delta(\xi + eV) \propto -\rho_s(-eV) = \left. \frac{E}{\sqrt{E^2 - \Delta^2}} \right|_{E=V} \tag{15.4}$$

so we have the Ivs. V and $\frac{\mathrm{d}I}{\mathrm{d}V}$vs. V plots:



15.2 Two superconductors

Now we can ask what happens if we replace the normal metal in the setup with a superconductor, i.e. a SIS′ setup. We notate the creation operators for the SC on the left with $c, c^\dagger$ and the phase parameter $\phi_!$ and on the right with $d, d^\dagger$ and ϕ_2. There are now two types of currents:

- Quasiparticle currents (fermionic excitations)
- Cooper pair currents

We have two superconducting gaps Δ_L, Δ_R. We will find that the sums and differences of the gaps both enter. Let's try to think about how we can get current/tunneling when we place two superconductors next to each other.

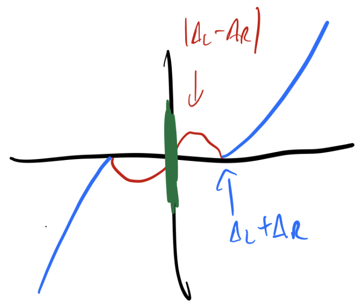
First consider the sum of the gaps term ($\Delta_L + \Delta_R$).

1. Applied voltage excites a pair from L-condensate. This costs $2\Delta_L$.
2. One fermion tunnels from left to right. This costs $\Delta_R - \Delta_L$.
3. The total energy of the process is $\Delta_L + \Delta_R$.

Now consider $T \neq 0$, where we can have a few thermally excited fermions. If $\Delta_R > \Delta_L$ for a thermally excited fermion to tunnel from L to R it costs $\Delta_R - \Delta_L$.

In the $\Delta_R < \Delta_L$ case, we have one thermally excited electron in R, and a pair broken out of the condensate by applying $2\Delta_L$ voltage. One of the two then tunnels to the right, costing $\Delta_R - \Delta_L$. Then, the pair in R goes into the condensate, so we get $-2\Delta_R$. The total energy cost is $\Delta_L - \Delta_R$.

We will see an IV-curve:



With

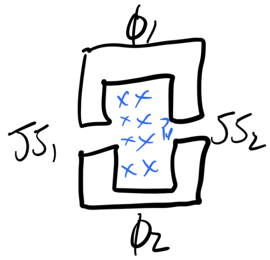
$$I = I_{qp} + I_J \tag{15.5}$$

and:

$$I_J = I_c \sin(\phi_1 - \phi_2 + 2evt) \tag{15.6}$$

where I_c is the critical current, which is very sensitive to H. If $I > I_c$ then there is no Josephson current.

A couple applications for Josephson effects/SQUIDs:



- SC qubits
- Scanning SQUID microscopy
- Magnetic encephalography
- Low field magnetometer
- Voltage standards

15.3 Hamiltonian formulation of tunneling (Cohen, Phillips, Falicov)

We consider a left/right superconductor, applying a voltage to the left. We then get:

$$H = H_L^0(V) + H_R^0 + H^T \tag{15.7}$$

with the tunnelling Hamiltonian:

$$H^T = \sum_{kk'} T_{kk'} \left[c_k^\dagger d_{k'} + d_{k'}^\dagger c_k \right] \tag{15.8}$$

We then have the number operator on the right superconductor:

$$N_R = \sum_{k'} d_{k'}^\dagger d_{k'} \tag{15.9}$$

Thus we get the current:

$$I = e\frac{\mathrm{d}N_R}{\mathrm{d}t} = ie[H^T, N_R] = i\sum_{kk'} T_{kk'}[c_k^\dagger d_{k'} - d_{k'}^\dagger c_k] \tag{15.10}$$

Introducing time dependence:

$$\left\langle \frac{\mathrm{d}N_R}{\mathrm{d}t} \right\rangle = -i\int_{t_0}^{t} dt' \left\langle [\frac{\mathrm{d}N_R(t)}{\mathrm{d}t}, H^T(t')] \right\rangle_0 \tag{15.11}$$

We will indeed find quasiparticle and Josephson terms emerging from this. Let's evaluate:

$$\left\langle \frac{\mathrm{d}N_R}{\mathrm{d}t} \right\rangle = \sum_{kk'} \int_{t_0}^{t} [c_k^\dagger(t) d_{k'}(t), d_{k'}^\dagger(t') c_k(t')] e^{ieV(t-t')} + \sum_{kk'} \int_{t_0}^{t} [c_k^\dagger(t) d_{k'}(t), c_{-k}^\dagger(t') d_{-k'}(t')] e^{-ieV(t+T')} \tag{15.12}$$

where the last term we can write as a time translation invariant piece $e^{ieV(t-t')}e^{-2ieVt}$. Schematically, the first term is $\left\langle c^\dagger c \right\rangle \left\langle d^\dagger d \right\rangle$ (these are the quasiparticle terms), which look like:

$$I_{qp} = (f(E_R) - f(E_L)) \left[\frac{1}{E_R - E_L + eV} - \frac{1}{E_R - E_L - eV} \right] + (1 - f(E_R) - f(E_L)) \left[\frac{1}{E_L + E_R + eV} - \frac{1}{E_R - E_L + eV} \right] \tag{15.13}$$

and the second term is $\left\langle c^\dagger c^\dagger \right\rangle \langle dd \rangle$. So the second anomalous term has dependence $\Delta_R \Delta_L e^{-2ieVt} = |\Delta_R||\Delta_L| e^{i(\theta_R - \theta_L)} e^{-2ieVt} \propto$, this is the Josephson term, and we get:

$$I_J \propto \sin(\theta_R - \theta_L + 2eVt) \tag{15.14}$$

16 Superconductivity in Unconventional Quantum Materials

Talk schedule for next Tuesday:

- Conrad: Topological Superconductivity
- Rio: Intro to Majoranas
- Nick: Superconductivity in Particle Physics
- Anish: Spintronics and superconductors

In this last lecture, we will discuss superconductivity in unconventional materials, which is not well understood by BCS. Pretty qualitative/survey-like discussion here, so I'm not going to take notes.