A Beginner's Guide to Majorana Fermions

PHYS 366 Final Presentation

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Outline

1. Historical background from particle physics
2. Majorana Fermions in Condensed Matter Systems
3. Quantum Computing with Majoranas
4. Experimental realization attempts

TEORIA SIMMETRICA DELL'ELETTRONE E DEL POSITRONE

Nota di Ettore Majorana

Sunto. - *Si dimostra la possibilità di pervenire a una piena simmetrizzazione formale della teoria quantistica dell'elettrone e del positrone facendo uso di un nuovo processo di quantizzazione. Il significato delle equazioni di* Dirac *ne risulta alquanto modificato e non vi è più luogo a parlare di stati di energia negativa; nè a presumere per ogni altro tipo di particelle, particolarmente neutre, l'esistenza di « antiparticelle » corrispondenti ai « vuoti » di energia negativa.*

Image Credit: Mondadori Collection

Majorana's Insight - Real Solutions

Solutions to the Dirac equation:

$$(i\gamma^{\mu}\partial_{\mu} - m)\Psi = 0 \tag{1}$$

Take the form:

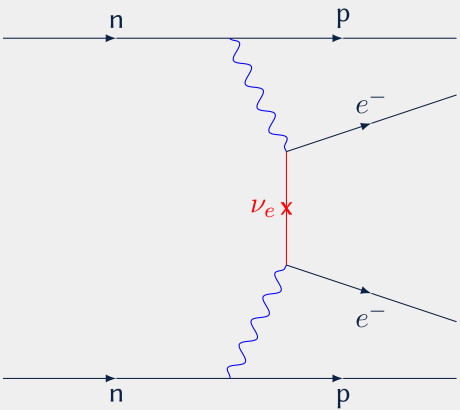
$$\Psi(x) = \underbrace{\sum_{E>0} a_E e^{-iEt}\Phi_E(x)}_{\text{particles}} + \underbrace{\sum_{E<0} b^{\dagger}_{-E} e^{-iEt}\Phi_E(x)}_{\text{anti-particles}} \tag{2}$$

Charge conjugation symmetry:

$$\Phi_{-E}(x) = \Phi^{c}(x) = C\Phi^{*}(x) \tag{3}$$

Enforcing $\Psi^{c}(x) = \Psi(x)$ (reality) - **Majorana fermion!** - with $b_E = a_E$.

Majoranas in Nature?



$\nu = \bar{\nu}$? Active research question.

Minimal model:

$$H = \int d^d r [\underbrace{h_0^{\sigma\sigma'}(\mathbf{r}) c^\dagger_{\sigma\mathbf{r}} c_{\sigma'\mathbf{r}}}_{\text{kinetic}} \underbrace{- V n_{\uparrow\mathbf{r}} n_{\downarrow\mathbf{r}}}_{\text{attraction}}] \tag{4}$$

Self-consistent mean-field decoupling:

$$-n_\uparrow n_\downarrow \approx \langle c^\dagger_\uparrow c^\dagger_\downarrow \rangle c_\uparrow c_\downarrow + c^\dagger_\uparrow c^\dagger_\downarrow \langle c_\uparrow c_\downarrow \rangle - \langle c^\dagger_\uparrow c^\dagger_\downarrow \rangle \langle c_\uparrow c_\downarrow \rangle \tag{5}$$

to get:

$$H_{\text{BdG}} = \int d^d r \left[h_0^{\sigma\sigma'}(\mathbf{r}) c^\dagger_{\sigma\mathbf{r}} c_{\sigma'\mathbf{r}} + (\Delta(\mathbf{r}) c^\dagger_{\uparrow\mathbf{r}} c^\dagger_{\downarrow\mathbf{r}} + h.c.) \right] - \frac{1}{V} |\Delta(\mathbf{r})|^2 \tag{6}$$

with (spatially varying) SC order parameter:

$$\Delta(\mathbf{r}) = V \langle c_{\uparrow\mathbf{r}} c_{\downarrow\mathbf{r}} \rangle \tag{7}$$

Defining Nambu spinor and h:

$$\Psi_{\mathbf{r}} = \begin{pmatrix} c_{\uparrow\mathbf{r}} \\ c_{\downarrow\mathbf{r}} \\ c^\dagger_{\downarrow\mathbf{r}} \\ -c^\dagger_{\uparrow\mathbf{r}} \end{pmatrix} = \begin{pmatrix} \psi_{\mathbf{r}} \\ i\sigma^y\psi^*_{\mathbf{r}} \end{pmatrix}, \quad h_{\text{BdG}}(\mathbf{r}) = \begin{pmatrix} h_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -\sigma^y h_0^*(\mathbf{r})\sigma^y \end{pmatrix} \tag{8}$$

we get:

$$\boxed{H_{\text{BdG}} = \int d^d r \left[\Psi^\dagger_{\mathbf{r}} h_{\text{BdG}}(\mathbf{r})\Psi_{\mathbf{r}} - \frac{1}{V}|\Delta(\mathbf{r})|^2\right]} \tag{9}$$

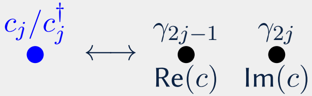
with Majorana(!!) fermion:

$$\boxed{C\Psi^*_{\mathbf{r}} := \tau^y\sigma^y\Psi^*_{\mathbf{r}} = \Psi_{\mathbf{r}}} \tag{10}$$

So superconductors admit natural description in terms of Majoranas. Also....

- $\Delta L = 2$ and $c^\dagger c^\dagger$ operators
- C-symmetry $\approx$ screening/confining $\mathbf{E}/\mathbf{H}$-fields
- Bogoliobov transformations $\approx$ Majorana transformations

Looking closer - Majorana Fermions as "half" of electrons



Fermions/electrons with (anti)-canonical commutation relations:

$$\{c_i^\dagger, c_j^\dagger\} = \{c_i, c_j\} = 0, \quad \{c_i^\dagger, c_j\} = \delta_{ij} \tag{11}$$

Majorana operators:

$$\gamma_{2j-1} = c_j^\dagger + c_j, \quad \gamma_{2j} = i(c_j^\dagger - c_j) \tag{12}$$

with algebra:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \boxed{\gamma_i^\dagger = \gamma_i} \tag{13}$$

If γ obeys:

$$[H, \gamma] = 0 \tag{14}$$

γs have zero energy:

$$H|E\rangle = E|E\rangle \implies H(i\gamma_1\gamma_2|E\rangle) = (i\gamma_1\gamma_2)H|E\rangle = E(i\gamma_1\gamma_2|E\rangle) \tag{15}$$

(Note, more physically):

$$[H, \gamma] \sim e^{-x/\xi} \tag{16}$$

Concrete Model: L spinless(!) fermions in 1-D:

$$H = \sum_j \left[-\underbrace{t(c_j^\dagger c_{j+1} + h.c.)}_{\text{hopping}} - \underbrace{\mu(c_j^\dagger c_j - \frac{1}{2})}_{\text{single-site}} + \underbrace{(\Delta c_j^\dagger c_{j+1}^\dagger + h.c.)}_{\text{superconductor}} \right] \tag{17}$$

In terms of $2L$ Majoranas:

$$H = \frac{i}{2} \sum_j [-\mu \gamma_{2j-1}\gamma_{2j} + (t + |\Delta|)\gamma_{2j}\gamma_{2j+1} + (-t + |\Delta|)\gamma_{2j-1}\gamma_{2j+1}] \tag{18}$$

Kitaev's Toy Model - trivial point ($|\Delta| = t = 0$)

$$H = \frac{i}{2}(-\mu)\sum_j \gamma_{2j-1}\gamma_{2j} = -\mu\sum_{j=1}^{L}\left(c_j^\dagger c_j - \frac{1}{2}\right) \tag{19}$$

Just L non-interacting electrons; nothing to see here.

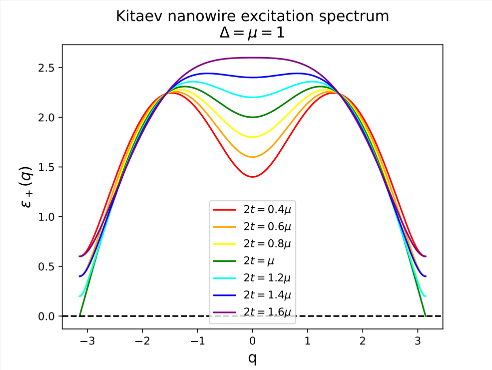
Kitaev's Toy Model - topological point ($|\Delta| = t > 0, \mu = 0$)

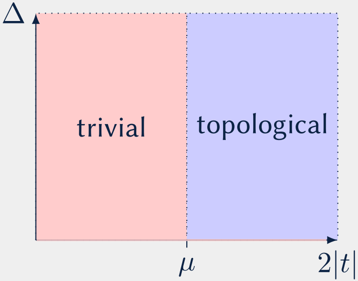
$$H = it \sum_j \gamma_{2j}\gamma_{2j+1} = 2t \sum_{j=1}^{L-1} \left(\tilde{c}_j^\dagger \tilde{c}_j - \frac{1}{2} \right) \tag{20}$$

γ_1, γ_{2L} are MZMs - we get zero energy excitations at the edge!

Spectrum:

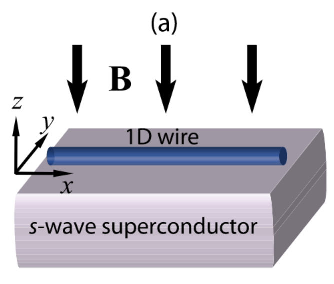
$$\epsilon_{\pm}(q) = \pm\sqrt{(2t\cos q + \mu)^2 + 4|\Delta|^2 \sin^2 q} \tag{21}$$





BUT - unrealistic - spinless (electrons have spin!) and long-range ordered (Mermin-Wagner)

How do we realistically realizing Majoranas? - Realistic 1D system



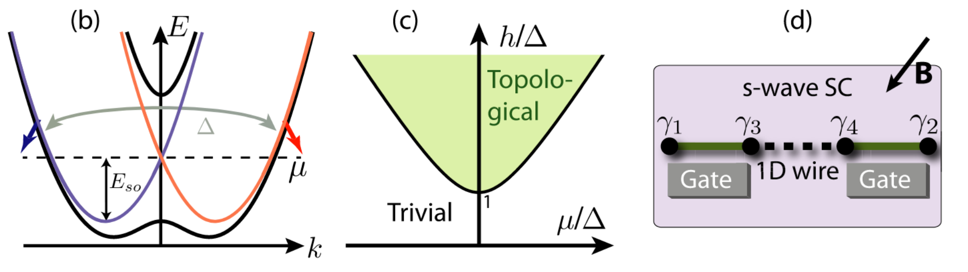
$$H = H_{\text{wire}} + H_{\Delta} \tag{22}$$

$$H_{\text{wire}} = \sum_{\sigma\sigma'} \int dx \psi_{\sigma}^{\dagger} \left(\underbrace{-\frac{\partial_x^2}{2m}}_{\text{kinetic}} - \mu + \underbrace{i\alpha\sigma^y\partial_x}_{\text{spin-orbit}} + \underbrace{h\sigma^z}_{\text{Zeeman}} \right) \psi_{\sigma'} \tag{23}$$

$$H_{\Delta} = \int dx (\Delta\psi_{\uparrow}^{\dagger}\psi_{\downarrow}^{\dagger} + \Delta^{*}\psi_{\uparrow}\psi_{\downarrow}) \tag{24}$$

Image Credit: J. Alicea 2012 Rep.Prog.Phys. 75 076501

Spin-Orbit, Zeeman, and Proximity to the Rescue



1. Two parabolas due to $\pm$ spin-couplings (blue/red); h then breaks the symmetry, creating the gap (black) - wire then appears spinless(!).
2. Superconducting proximity effect allows for lower-band electrons to p-wave pair, driving state to long-ranged(!) topological superconductor, so long as:

$$h > \sqrt{\Delta^2 + \mu^2} \tag{25}$$

Image Credit: J. Alicea 2012 Rep.Prog.Phys. 75 076501

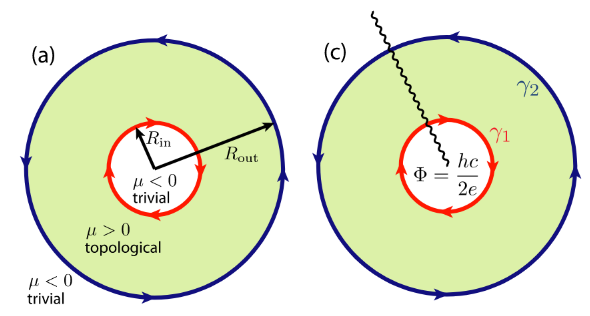
Other roads to Majorana zero modes

Other 1D realizations:

- Edges of 2D topological insulators
- Wires of 3D topological insulators

2D:

- Spinless $p + ip$ spinless superconductors
- FQH states
- Intrinsic $p + ip$ superconductivity
- 3D topological insulators

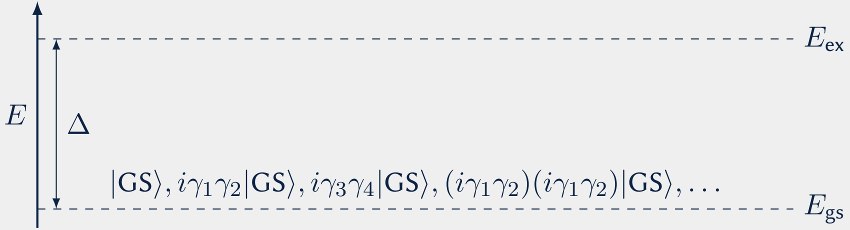


$$H = \int d^2r \left[\psi^\dagger \left(-\frac{\nabla^2}{2m} - \mu \right) \psi + \frac{\Delta}{2} [e^{i\phi} \psi \underbrace{(\partial_x + i\partial_y)}_{p_x + ip_y \text{pairing}} \psi + h.c.] \right] \quad (26)$$

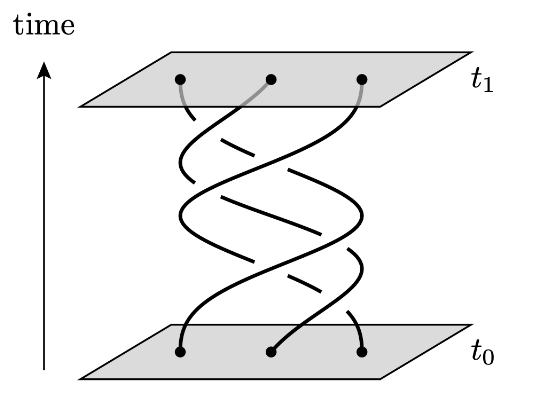
- Candidate: Sr_2RuO_4 (Intrinsic $p + ip$), hosts vortices (Majorana binding sites).
- Complications; $p_x \pm ip_y$ degeneracy, $E_{\text{vortex}} \sim \frac{(k_F \Delta)^2}{E_F} \sim$ mK

Image Credit: J. Alicea 2012 Rep.Prog.Phys. 75 076501

MZMs $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \dots \gamma_{2N-1}, \gamma_{2N}$ yields exponential GSD



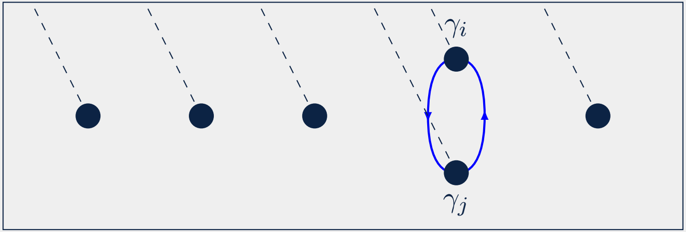
- Topologically protected encoding into eigenstates of $n_j = \frac{1}{2}(1 + i\gamma_{2j-1}\gamma_{2j})$.
- Topologically protected evolution via braiding.



- 3D: $\Pi_{\text{ex}}^2 = 1 \implies$ bosons, fermions.
- 2D: “Anyons” with nontrivial exchange statistics.

Image credit: S. Burton, arXiv:1610.05384v1

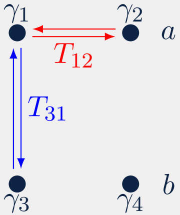
$\Delta \to e^{i\phi}\Delta$ results in $c_a \to e^{i\phi/2}c_a$ and $c_a^\dagger \to e^{-i\phi/2}c_a^\dagger$. So for $\phi = 2\pi$, $\gamma \to -\gamma$.
"Cut" picture for exchange T_{ij}:



$$\gamma_i \to -\gamma_j, \quad \gamma_j \to \gamma_i, \quad \gamma_k \to \gamma_k \tag{27}$$

Or as an operator:

$$T_{ij} = \frac{1}{\sqrt{2}}(1 + \gamma_j\gamma_i) \tag{28}$$



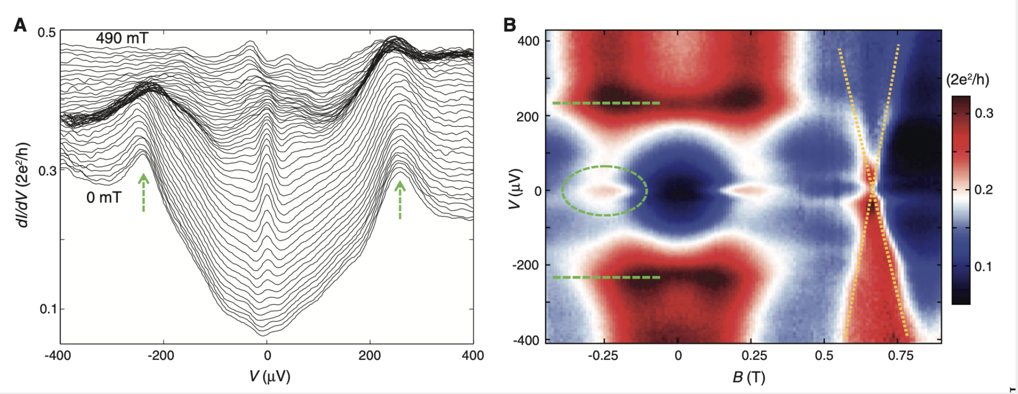
Basis states $|n_a, n_b\rangle$ with $n_a = \frac{1}{2}(1 + i\gamma_1\gamma_2)$ and $n_b = \frac{1}{2}(1 + i\gamma_3\gamma_4)$.

$$T_{12}|n_a, n_b\rangle = e^{i\frac{\pi}{4}(1-2n_a)}|n_a, n_b\rangle \tag{29}$$

$$T_{31}|n_a, n_b\rangle = \frac{1}{\sqrt{2}}[|n_a, n_b\rangle + (-1)^{n_a}|1 + n_a, 1 + n_b\rangle] \tag{30}$$

Takeaway:

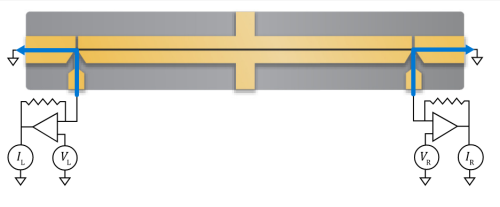
- Braiding $\implies$ topologically protected gateset.
- Note: $U_{\text{phase}} = \text{diag}(1, e^{i\Delta Et})$ (unprotected) needed for universality.

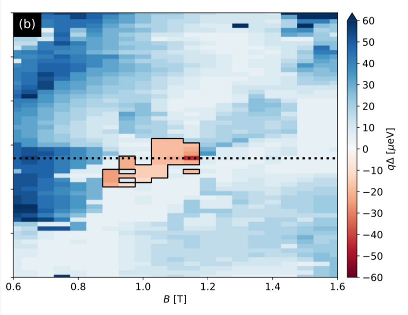


- Peak in $g(V) = \frac{dI}{dV} \propto \rho_{\text{wire-end}}$ can indicate MZMs, with $\frac{2e^2}{h}$ quantization.
- Disorder is also an explanation - stronger signatures needed.
- Microsoft 2018 retraction.

Image credit: V. Mourik et. al, Science 2012

Microsoft 2021-2023 - A more robust protocol?

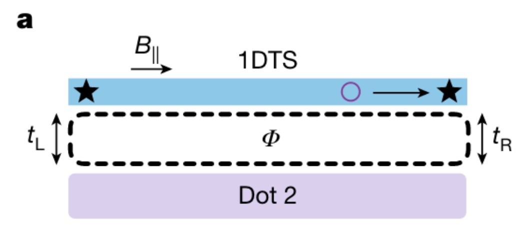


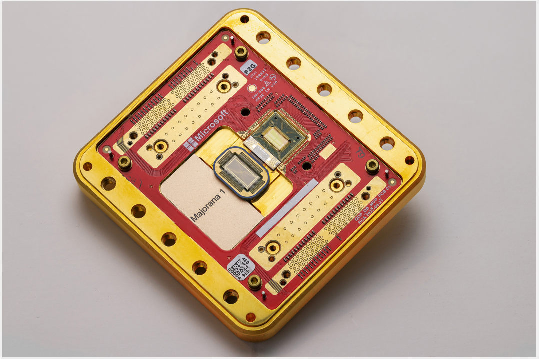


- Topological Gap Protocol (2021) - measurement of Δ_{bulk} closing via non-local conductance measurements $G_{LR}, G_{RL} = \frac{dI_L}{dV_R}, \frac{dI_R}{dV_L}$.
- (2023) Reports on devices passing the protocol - though with some controversy, e.g. in releasing parameters at publication, also a recent arXiv comment[1] suggests inconsistencies.

[1] arXiv:2502.19560v1

Image credit: Microsoft, Phys Rev. B 107, 245423 (2023)





"The editorial team wishes to point out that the results in this manuscript do not represent evidence for the presence of Majorana zero modes in the reported devices. The work is published for introducing a device architecture that might enable fusion experiments using future Majorana zero modes."

Image credit: Microsoft, Nature 2025 + Press Release

Tutorial Takeaways

- Original Conext + Importance in Particle theory
- Connection to superconductors
- MZMs in condensed matter systems
- How to harness MZMs for quantum computation
- Experimental signatures and progress

Thanks for listening!

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